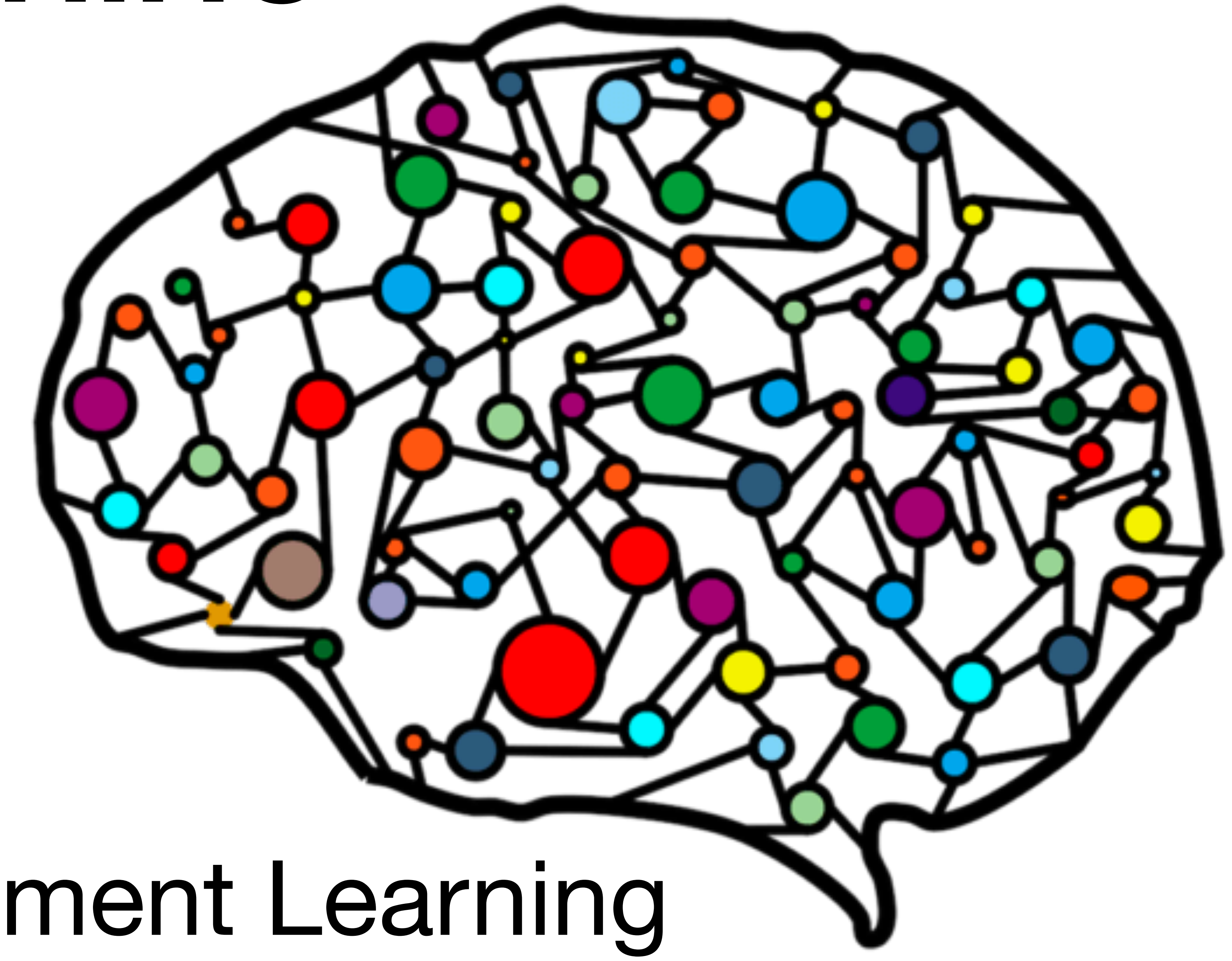
General Principles of Human and Machine Learning



Lecture 4: Introduction to Reinforcement Learning

Dr. Charley Wu

https://hmc-lab.com/GPHML.html

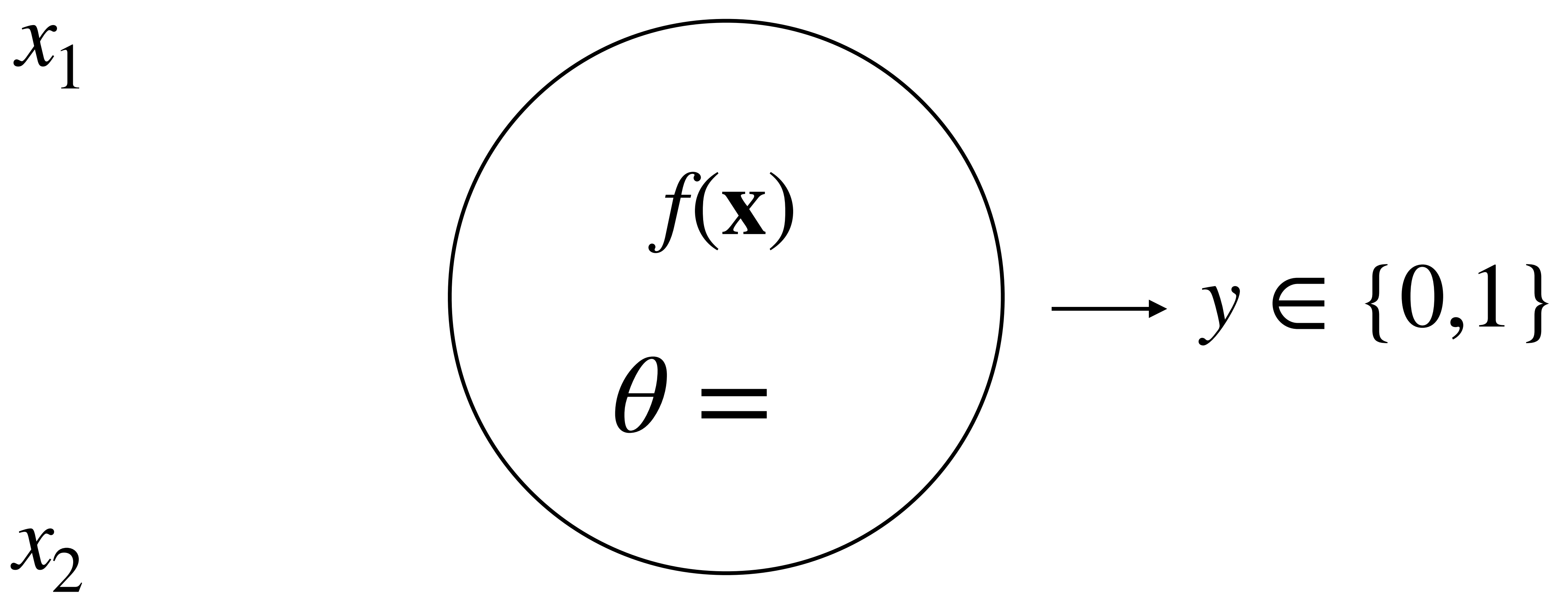
Quiz results

- Average grade 81%
- If you did well, please keep up the good work!
- If you wish you had done better, use this as a learning experience and remember that 1 pop quiz is a freebie (best 3 out of 4)
- If you missed the quiz but had a documented absence (email to me + TA 24hrs in advance), then we can work something out for further documented absences
- Quiz questions may reappear on the exam

Clarifications

NAND

$$f(\mathbf{x}) = \begin{cases} 1 & \text{if } \sum w_i x_i \geq \theta \\ 0 & \text{else} \end{cases}$$



Neuron fires when x_1 is on AND x_2 not on

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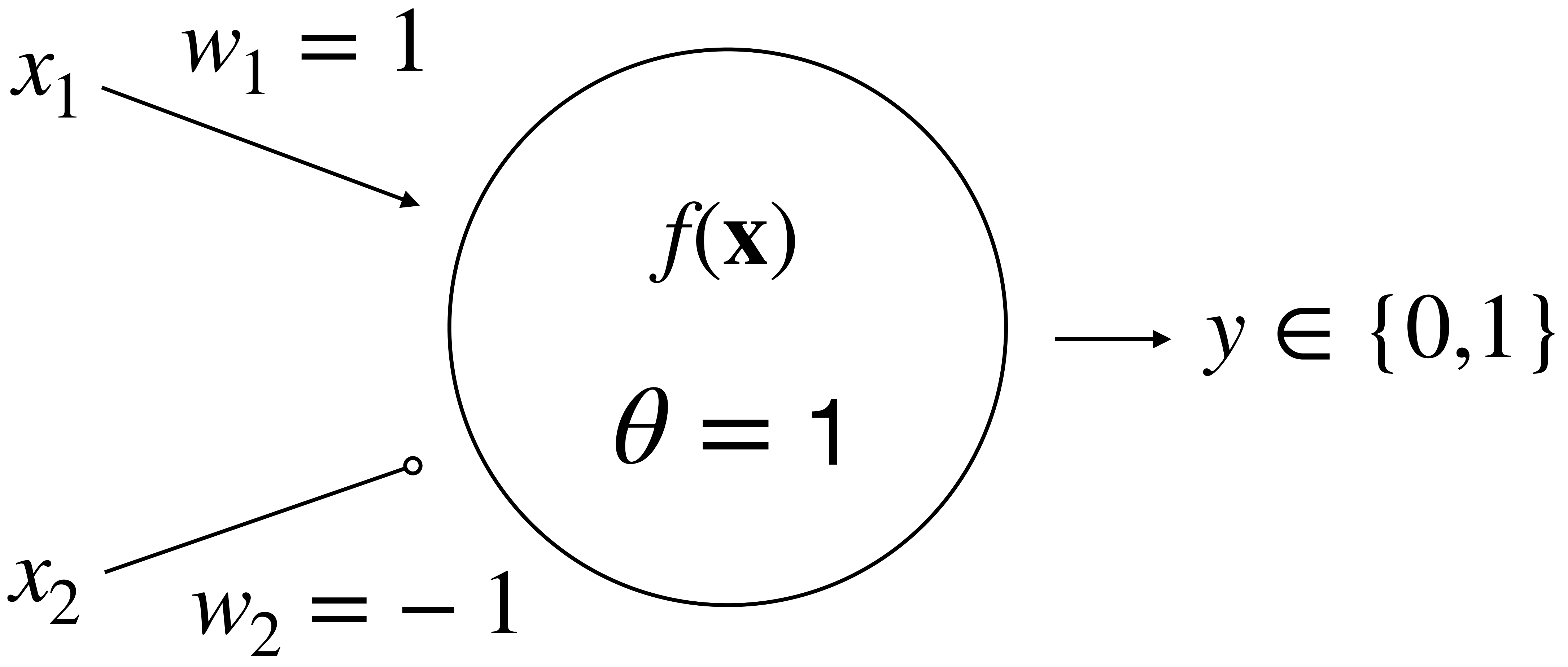

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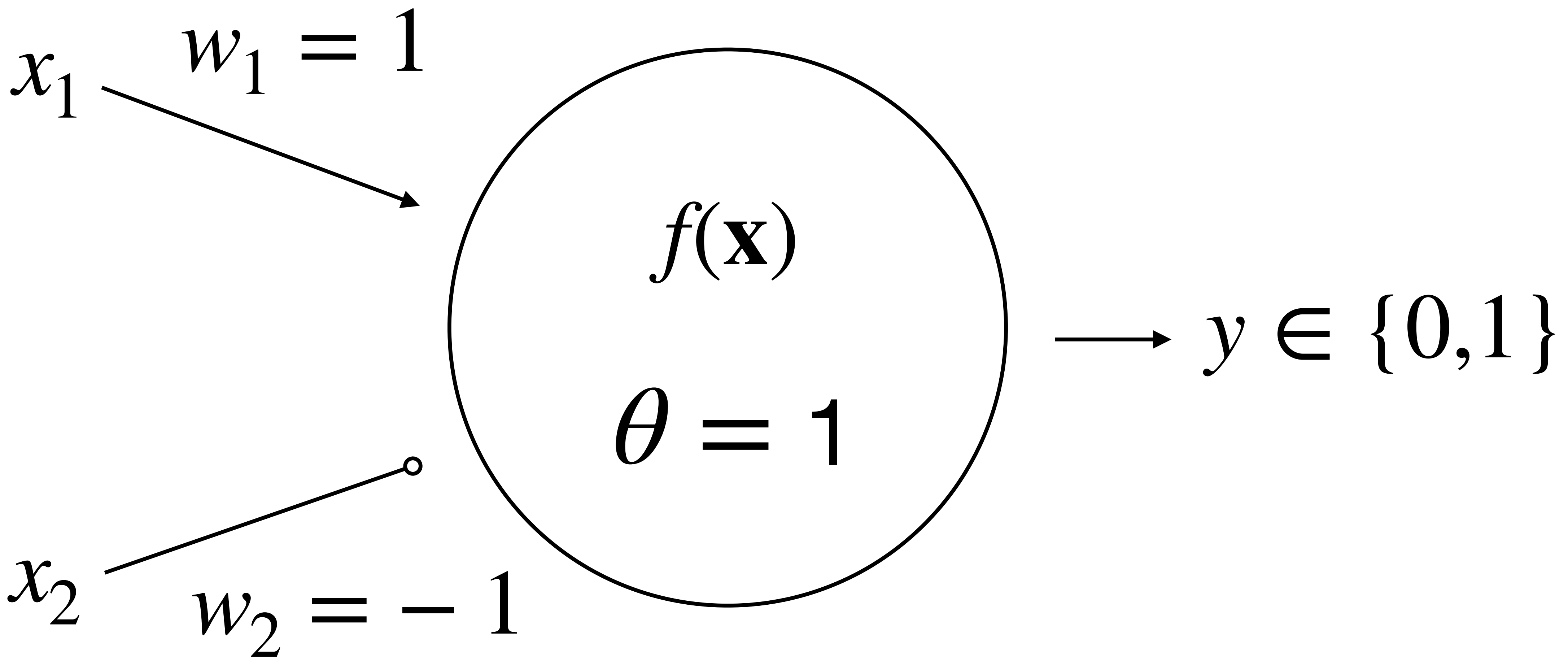

Neuron fires when x_1 is on AND x_2 not on

Rescorla-Wagner

Reward prediction

$$\hat{r}_t = \sum_i \mathrm{CS}_i^t w_i$$

Weight update

For i where $CS_i = 1$:

$$w_i \leftarrow w_i + \eta(r_t - \hat{r}_t)$$

Learning rate

Observed outcome

Predicted outcome

δ Reward prediction error (RPE)

Clarifications

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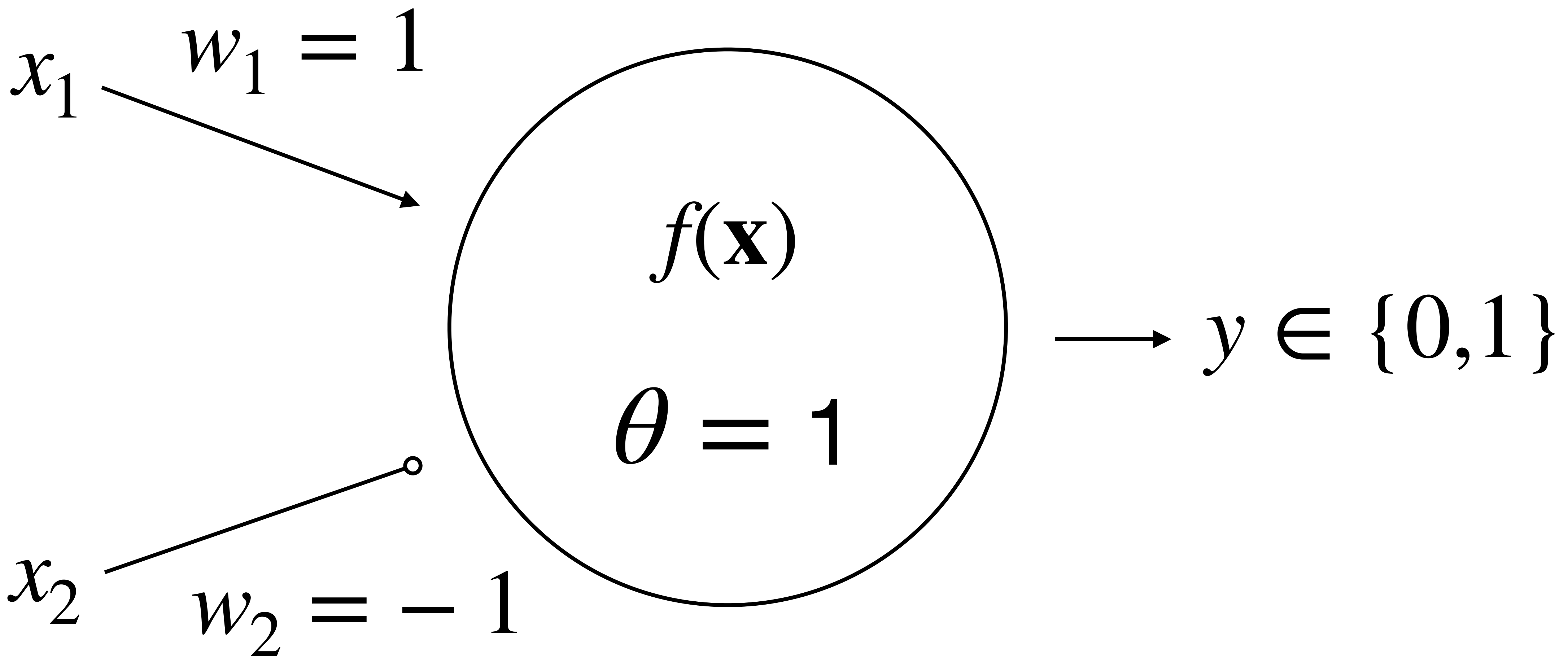

Neuron fires when x_1 is on AND x_2 not on

Rescorla-Wagner

Reward prediction

$$\hat{r}_t = \sum_i \mathrm{CS}_i^t w_i$$

Weight update

Larger when better reward than expected!

For i where $CS_i = 1$:

$$w_i \leftarrow w_i + \eta(r_t - \hat{r}_t)$$

Learning rate

Observed outcome

Predicted outcome

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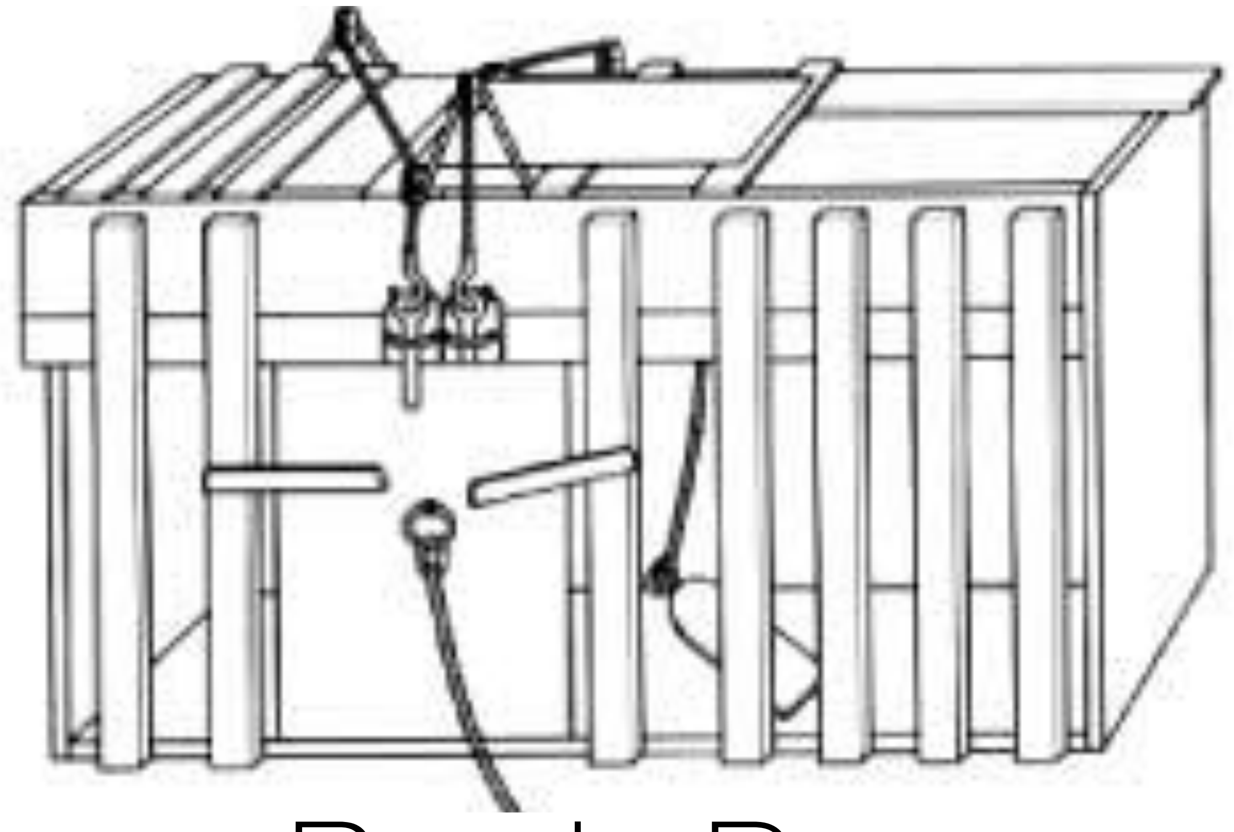
The story so far …

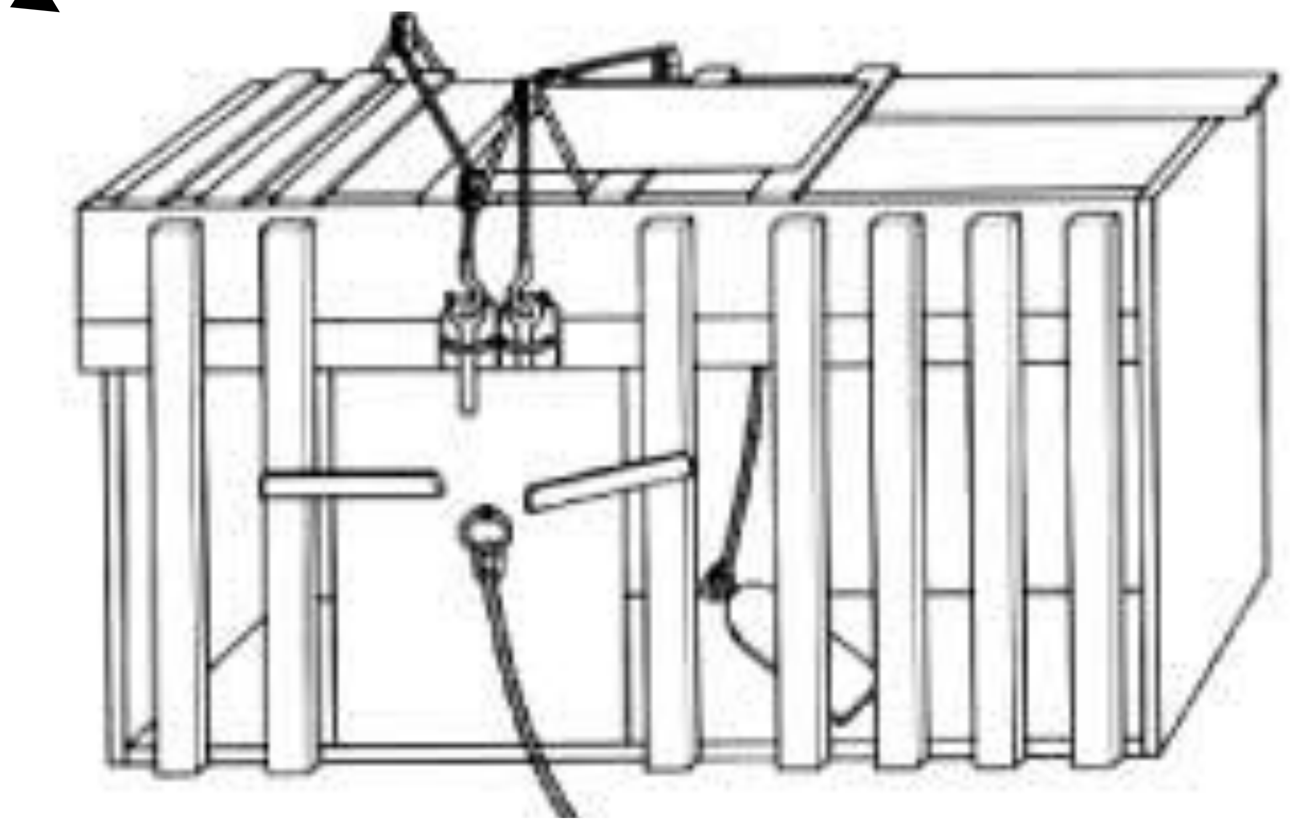
Thorndike's (1898) Law of Effect



Puzzle Box

Thorndike's (1898) Law of Effect

Cat



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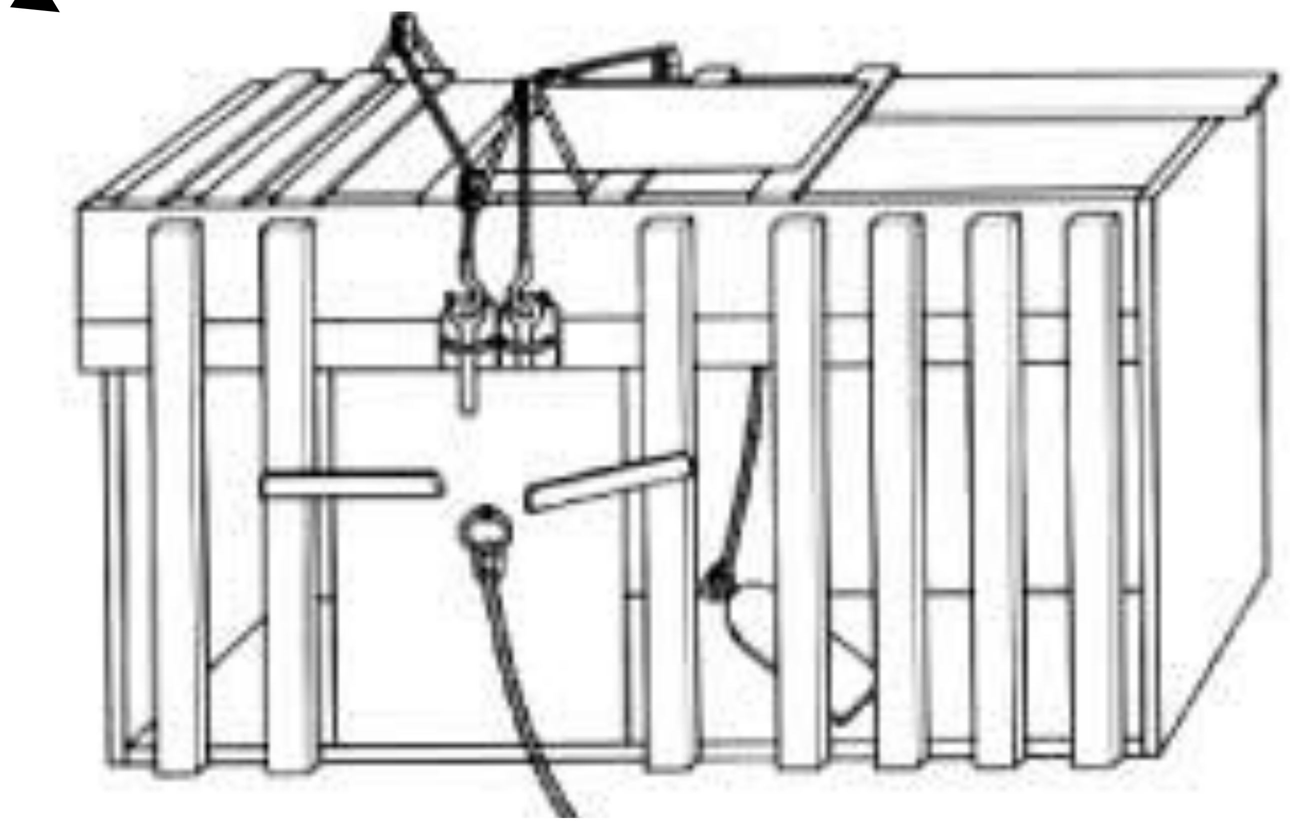

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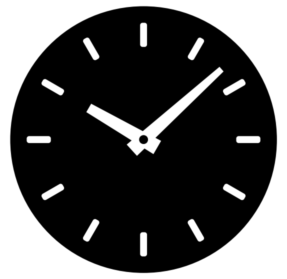
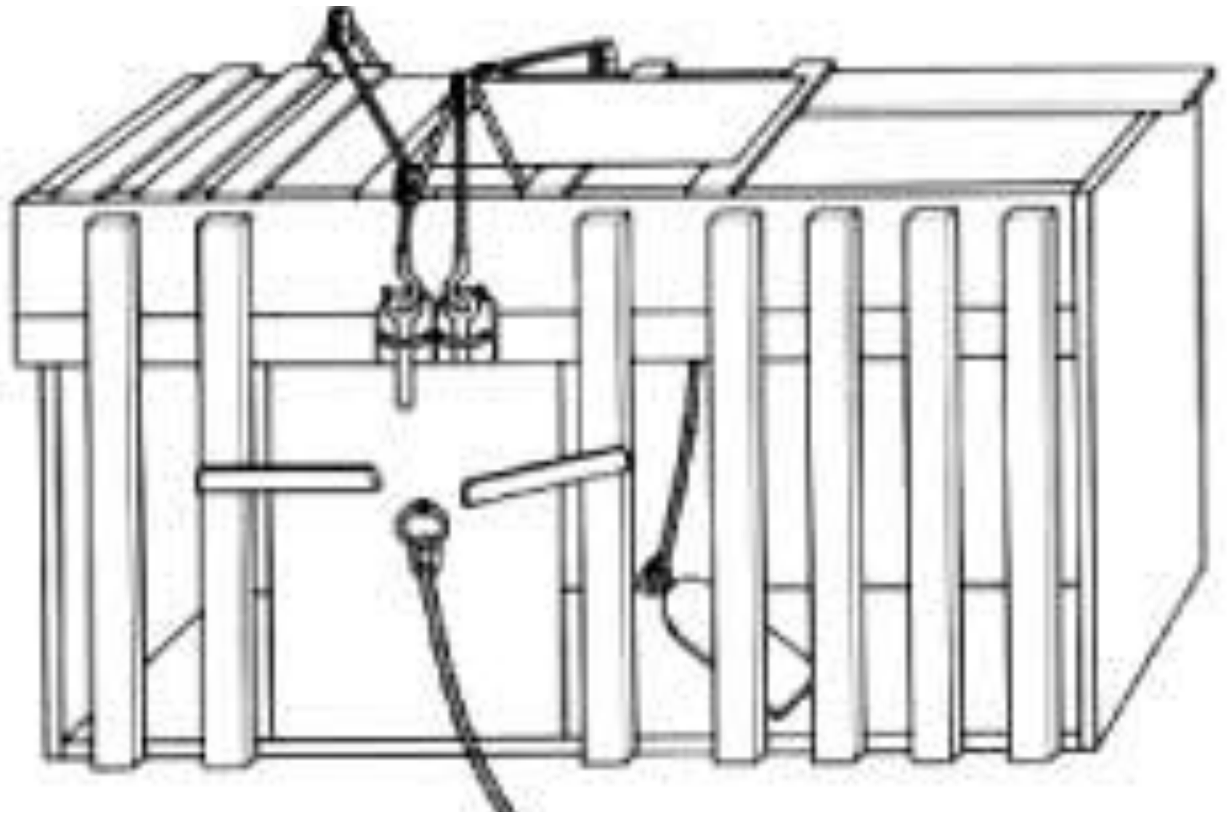


Time to escape

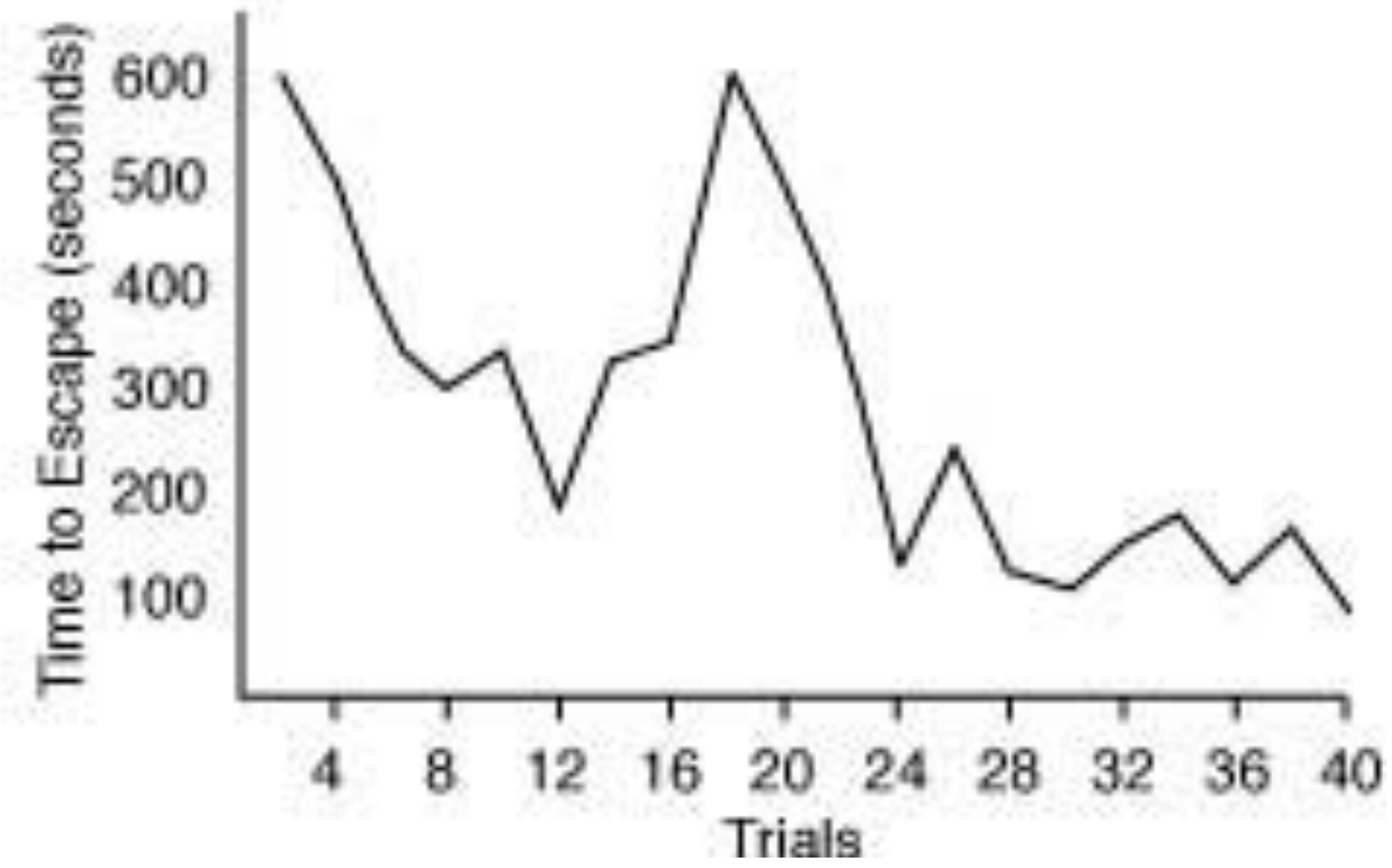
Thorndike's (1898) Law of Effect

Cat



Puzzle Box

Time to escape

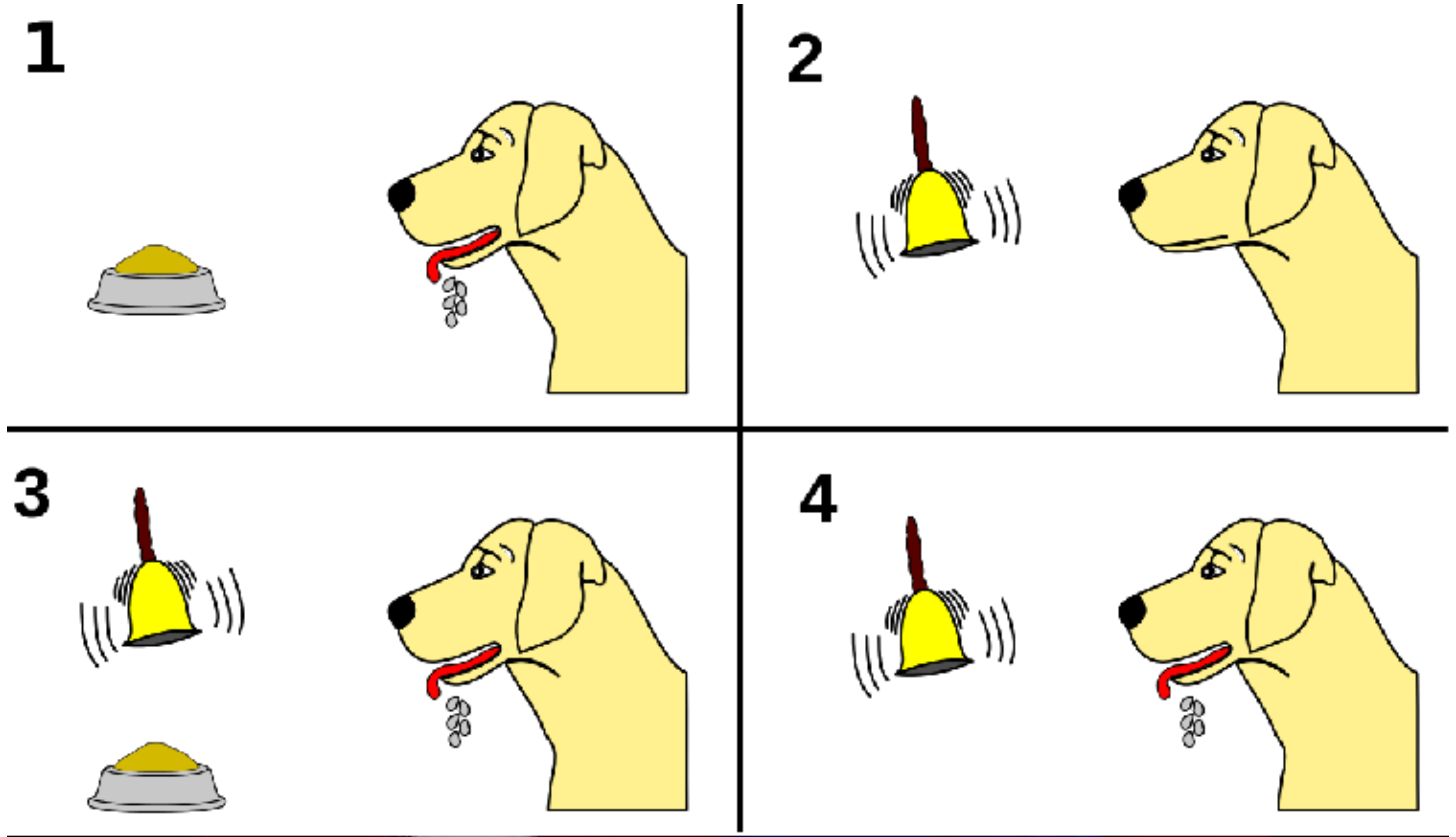


Actions associated with satisfaction are strengthened, while those associated with discomfort become weakened.

Classical and Operant Conditioning

Classical Condition (Pavlov, 1927)

Learning as the *passive* coupling of stimulus (bell ringing) and response (salivation), anticipating future rewards



Operant Condition (Skinner, 1938)

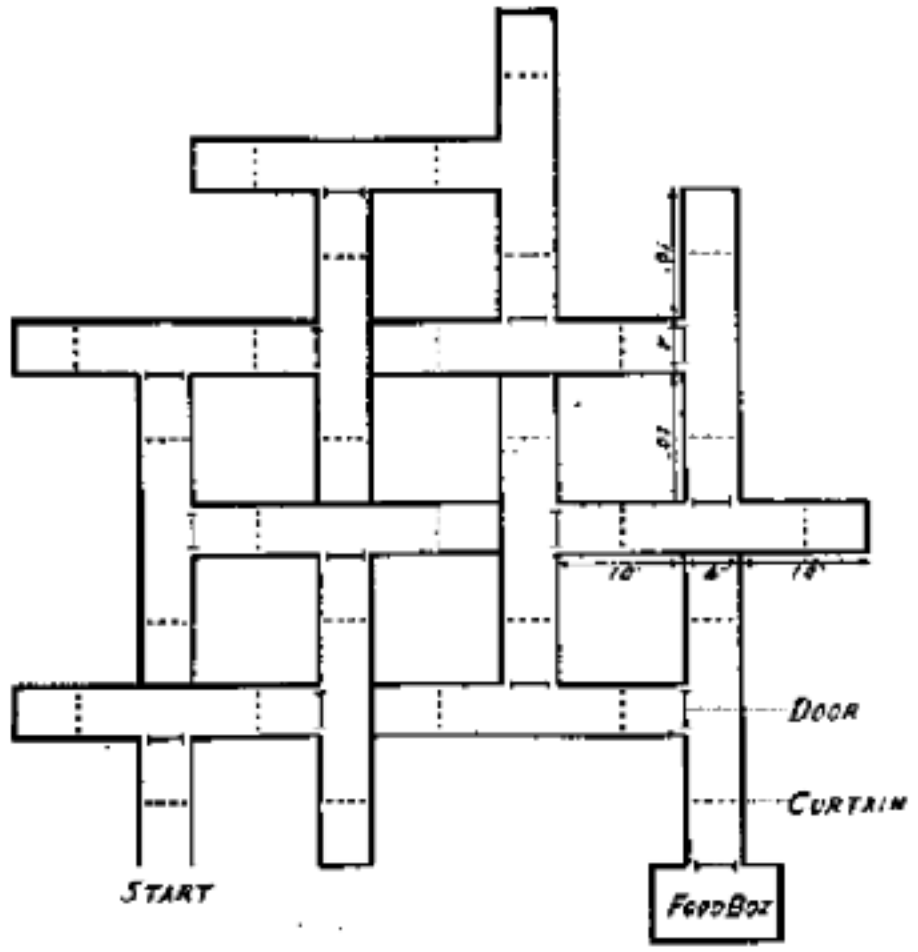
Skinner (1938): Learning as the *active* shaping of behavior in response to rewards or punishments

Tolman and Cognitive maps

- Learning is not just a telephone switchboard connecting incoming sensory signals to outgoing responses (S-R Learning)
- Rather, “latent learning” establishes something like a “field map of the environment” gets etablished (S-S learning)

Stimulus-Response (S-R) Learning

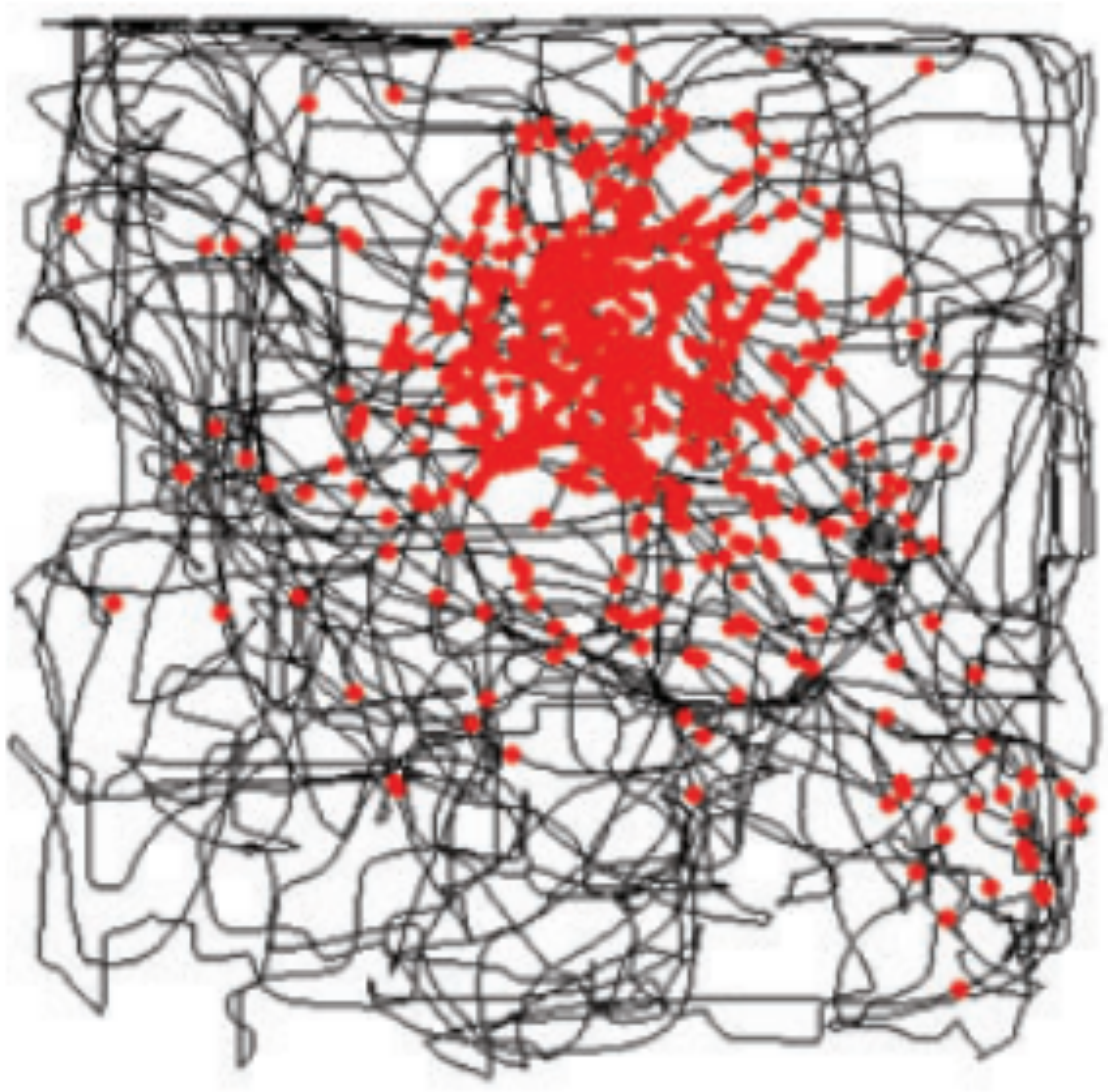
Stimulus-Stimulus (S-S) Learning



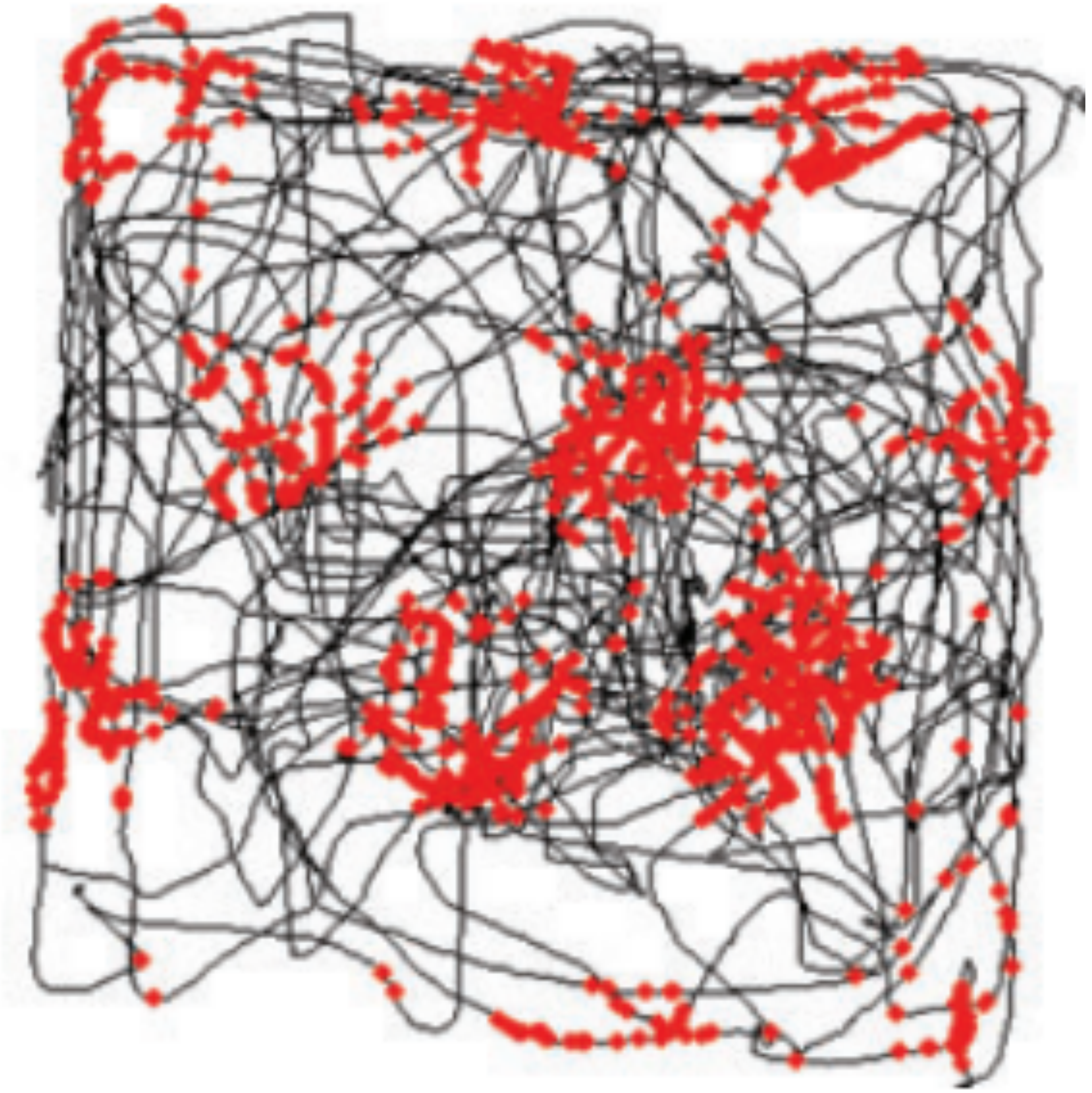
Plan of maze
14 Unit T Alley Maze
Fig. 1
(From M. H. Elliott, The effect of change of reward on the maze performance of rats. Univ. Calif. Publ. Psychol., 1928, 4, p. 20.)

Cognitive maps in biological brains

Place cells in the hippocampus

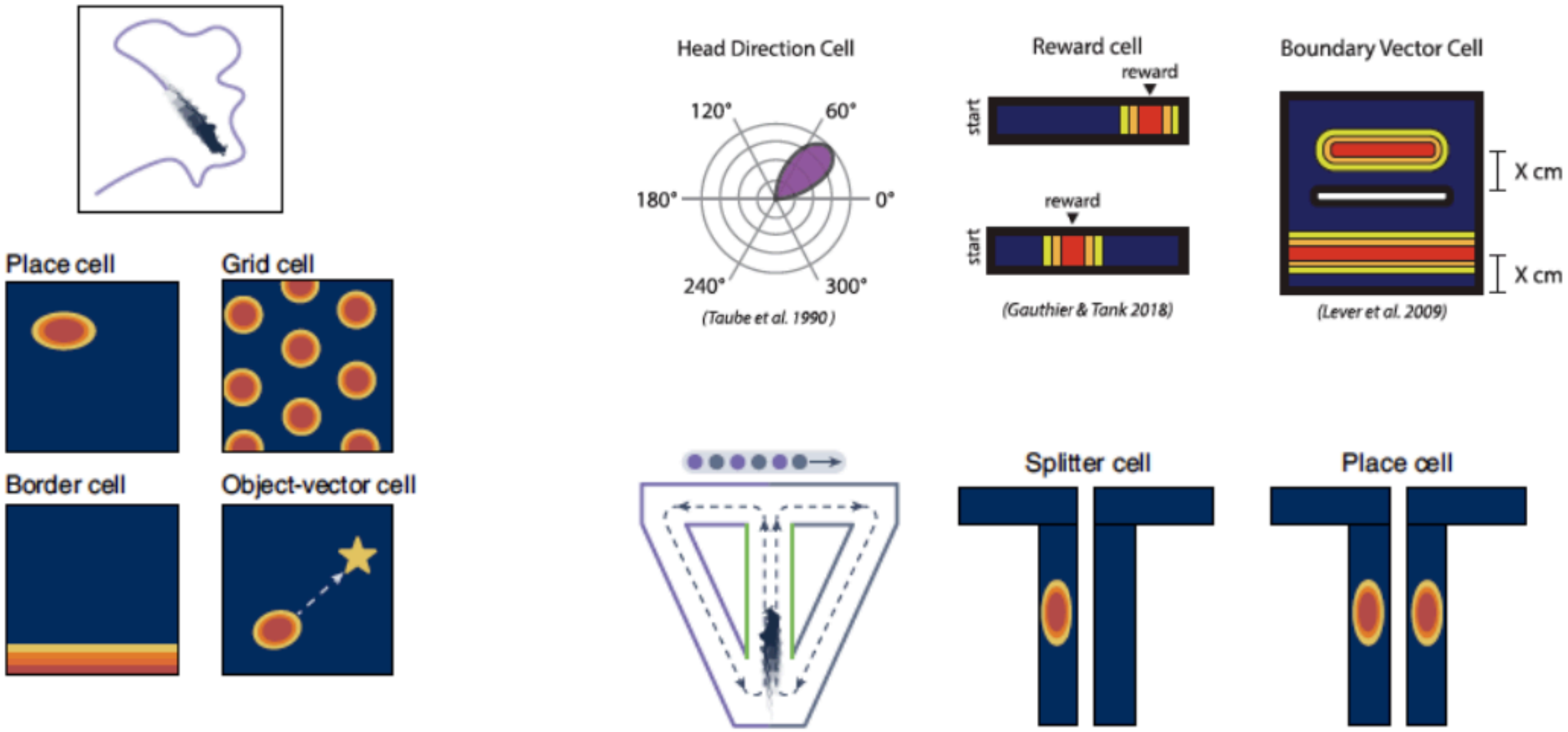


Grid cells in the medial entorhinal cortex



Moser et al., (*Ann Rev Neuro* 2008)

“Hippocampal zoo”

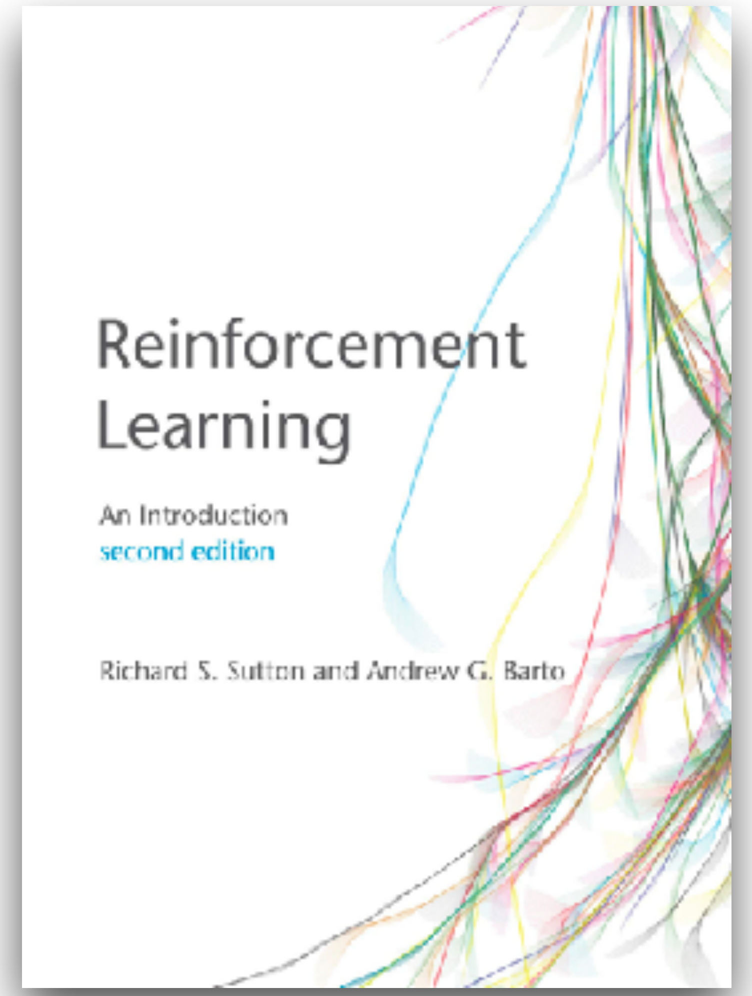


Behrens et al., (*Neuron* 2018)
Whittington et al,. (*Nat Neuro* 2022)

Agenda for today: From Tolman to Reinforcement Learning

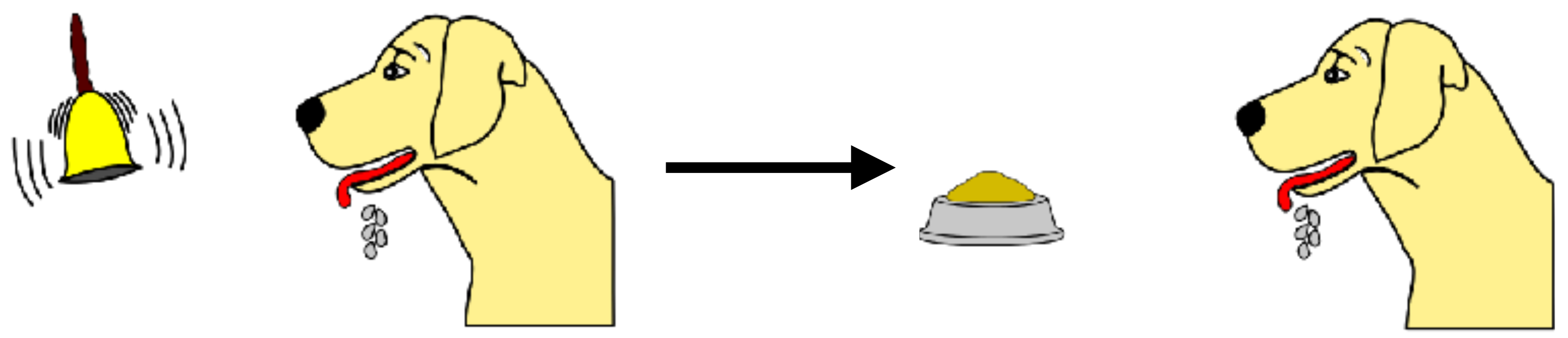
- **Part 1**: Introduce RL framework, origins, and terminology (Sutton & Barto)

- **Part 2:** Model-free vs. model-based RL

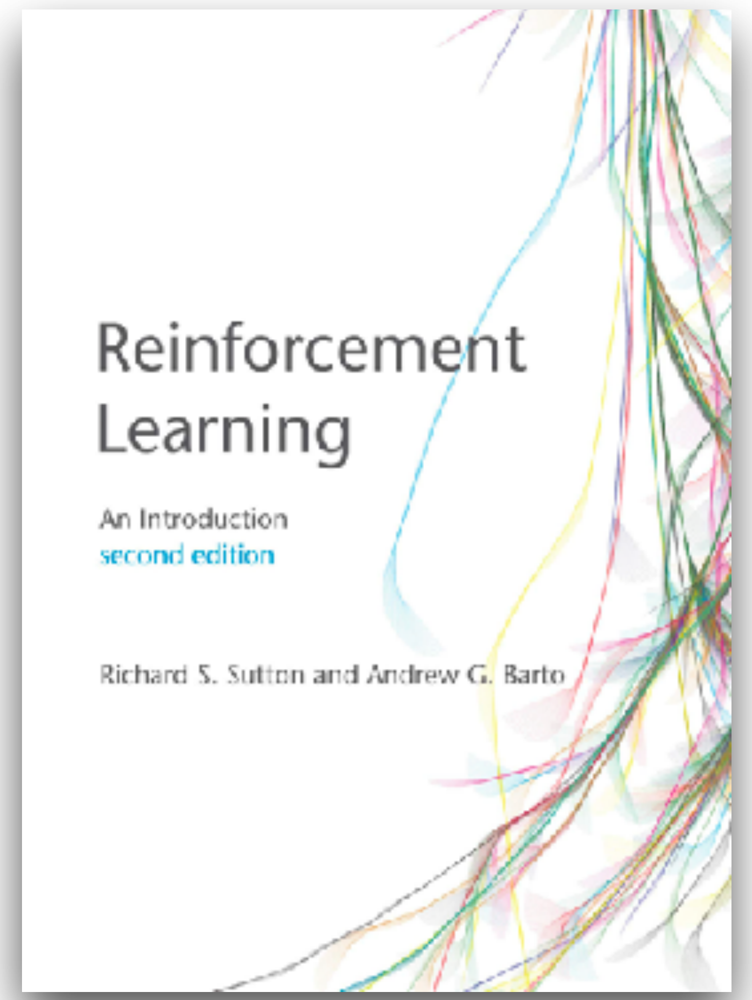


Reinforcement Learning

Pavlovian (classical) conditioning



Learn which environmental cues *predict* reward



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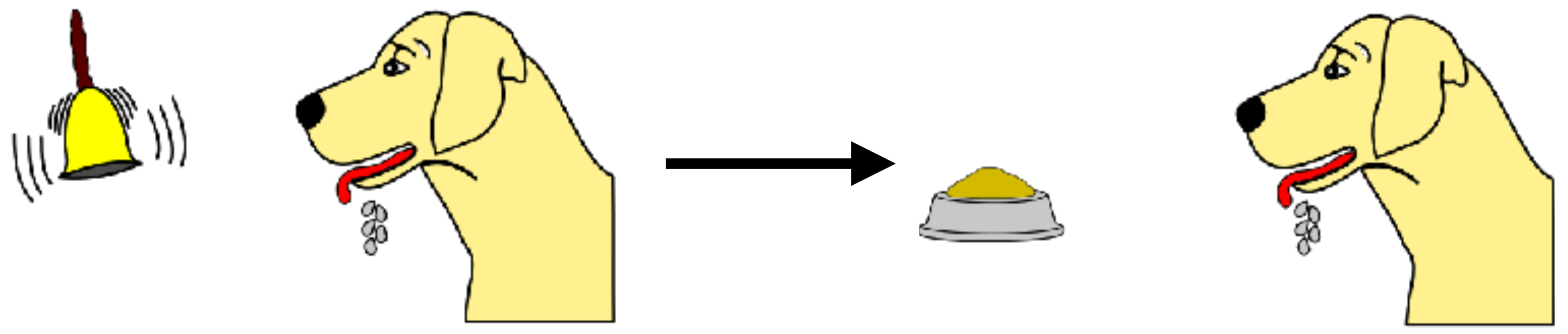

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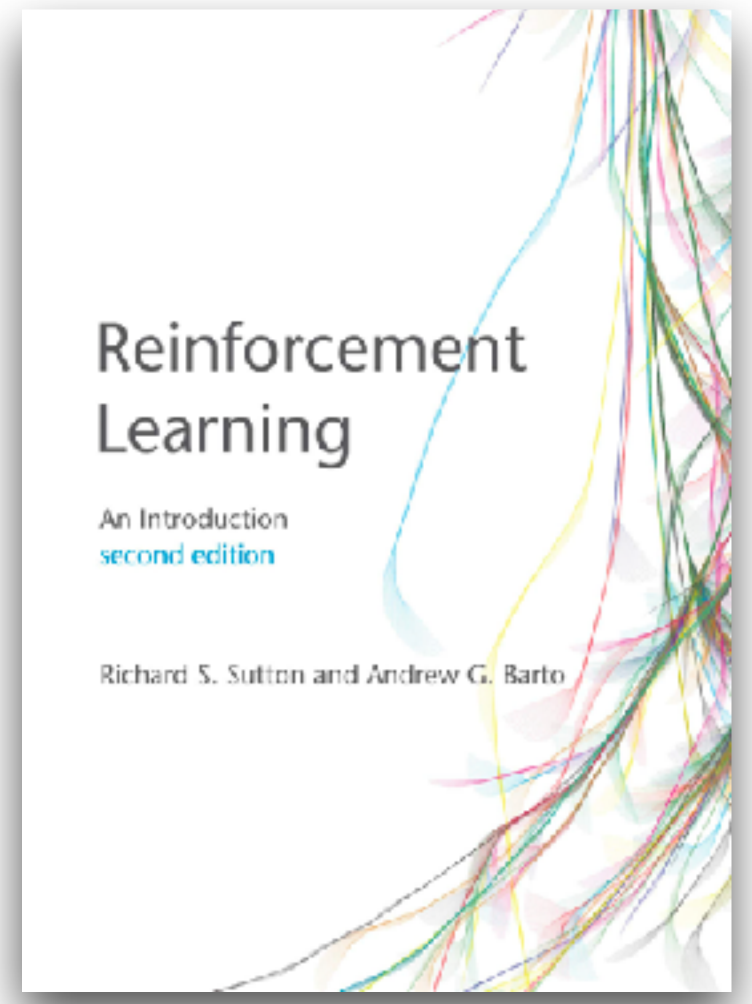
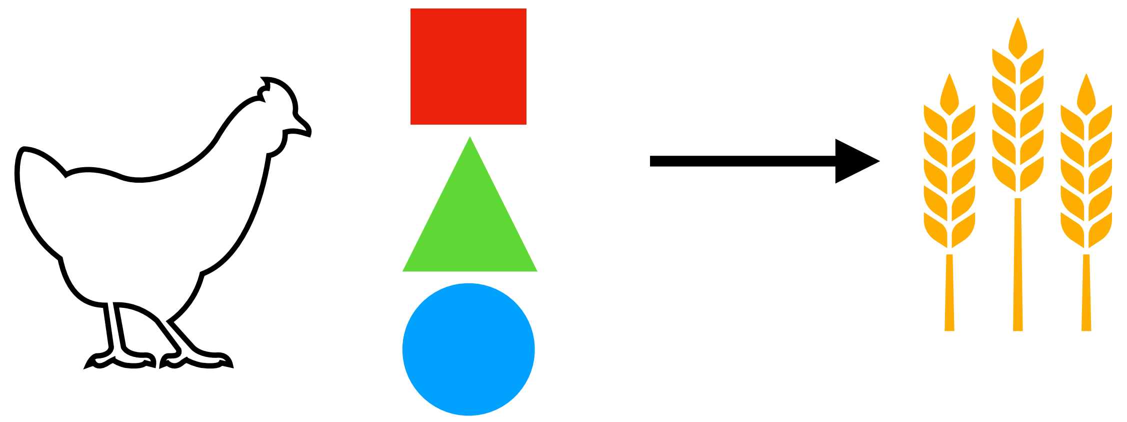


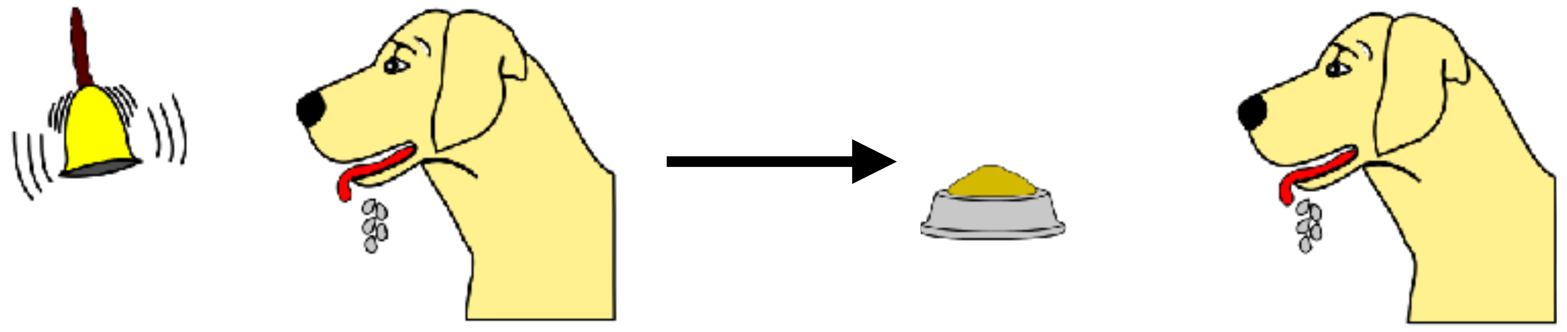
Reinforcement Learning

Operant (instrumental) conditioning



Learn which actions *predict* reward

Pavlovian (classical) conditioning



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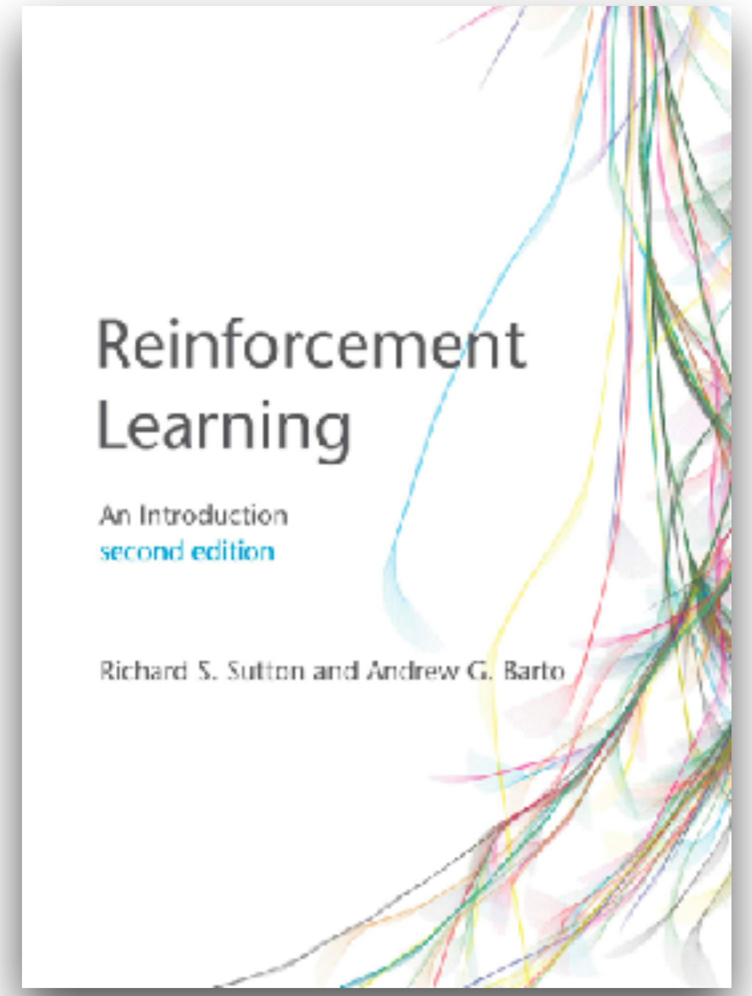

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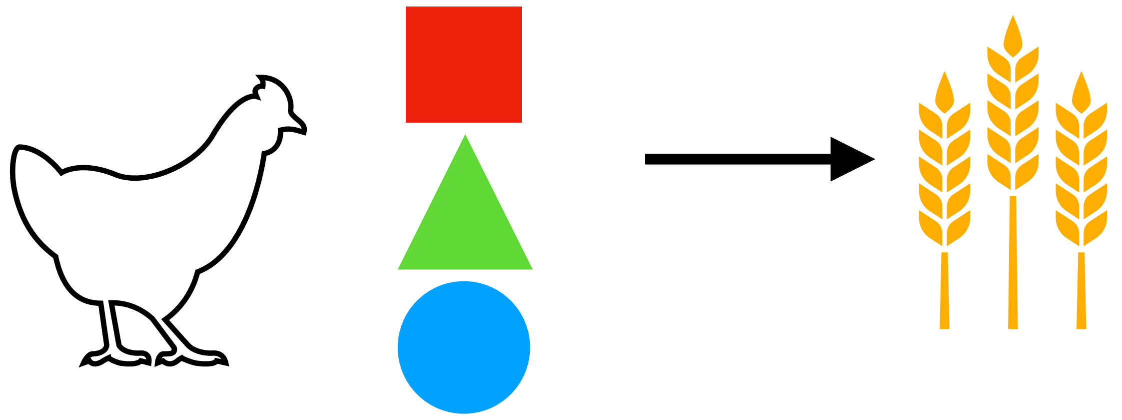
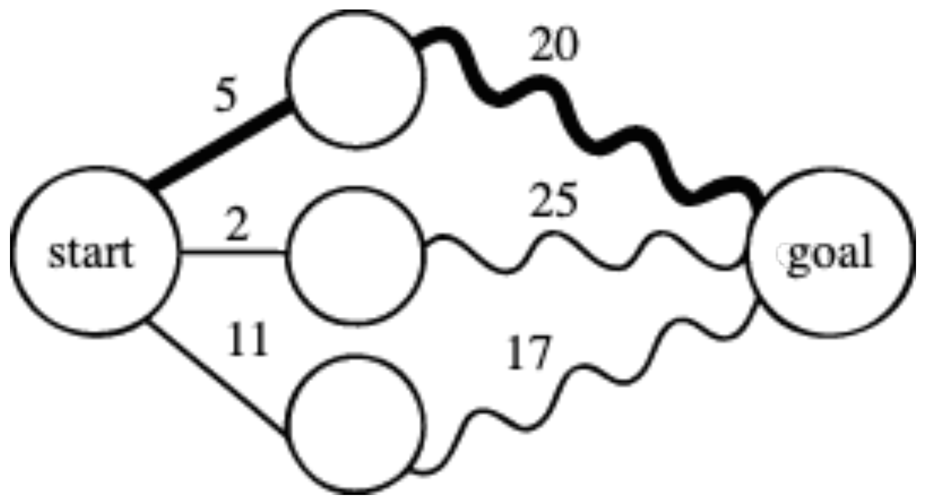


Learn which actions *predict* reward

Neuro-dynamic programing

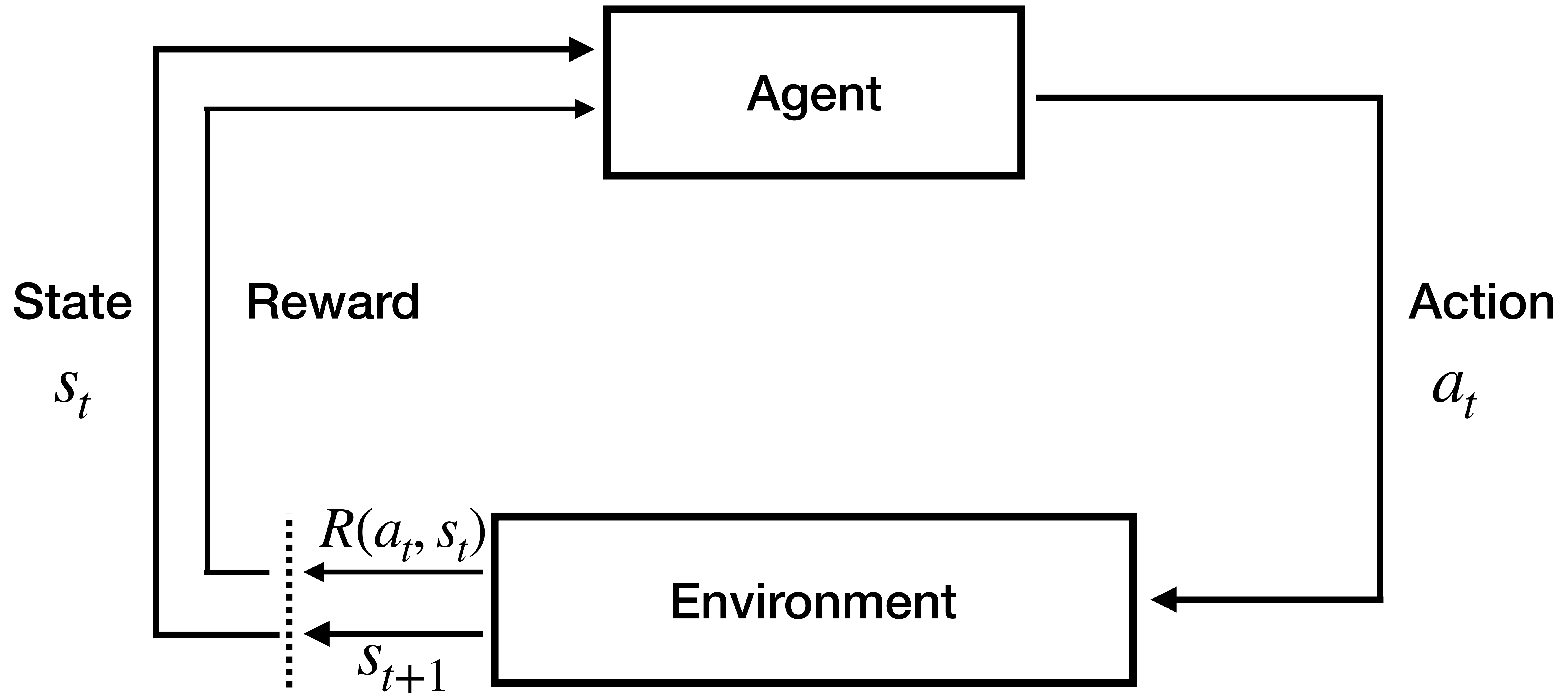
Bertsekas & Tsitsiklis (1996)

Stochastic approximations to dynamic programing problems



Sutton and Barto (2018 [1998])

Reinforcement Learning



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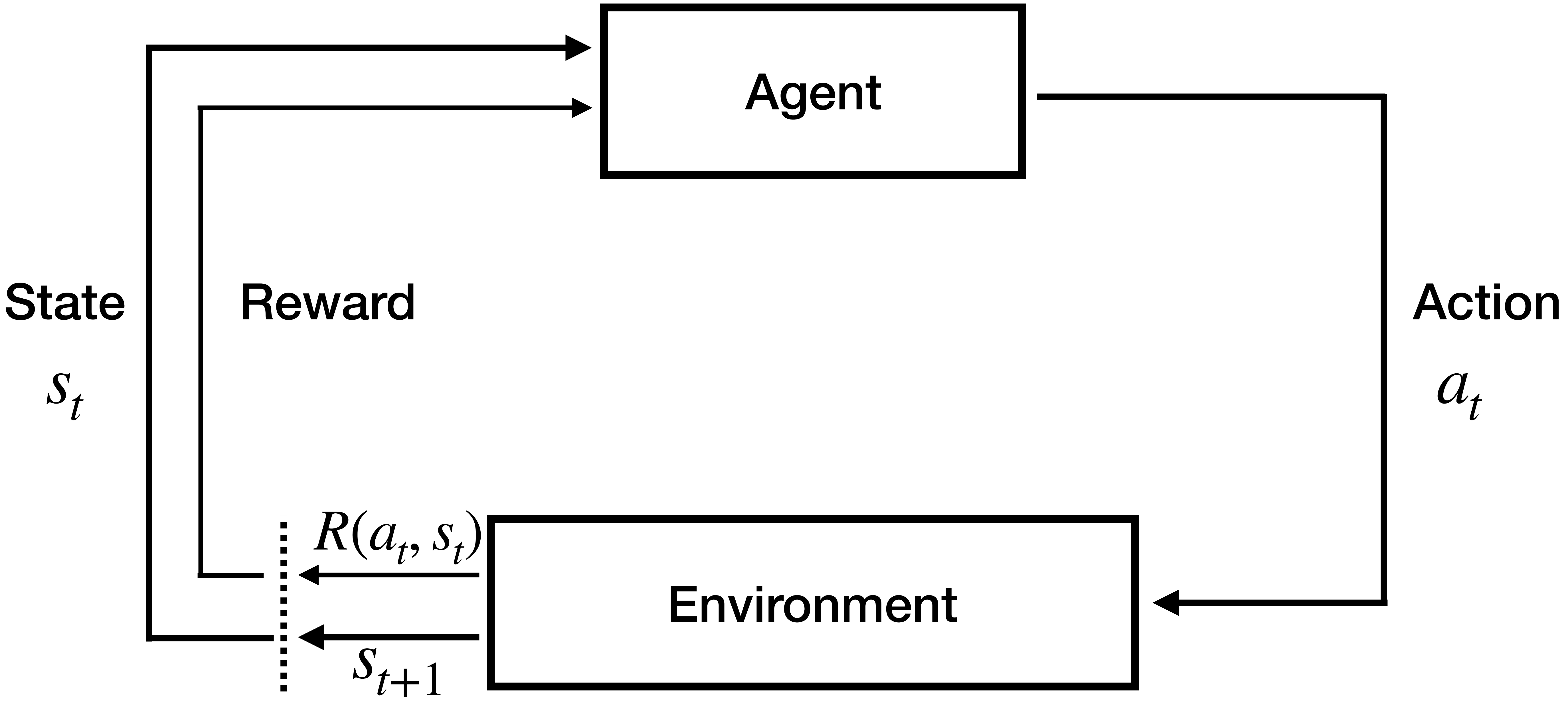
Reinforcement Learning

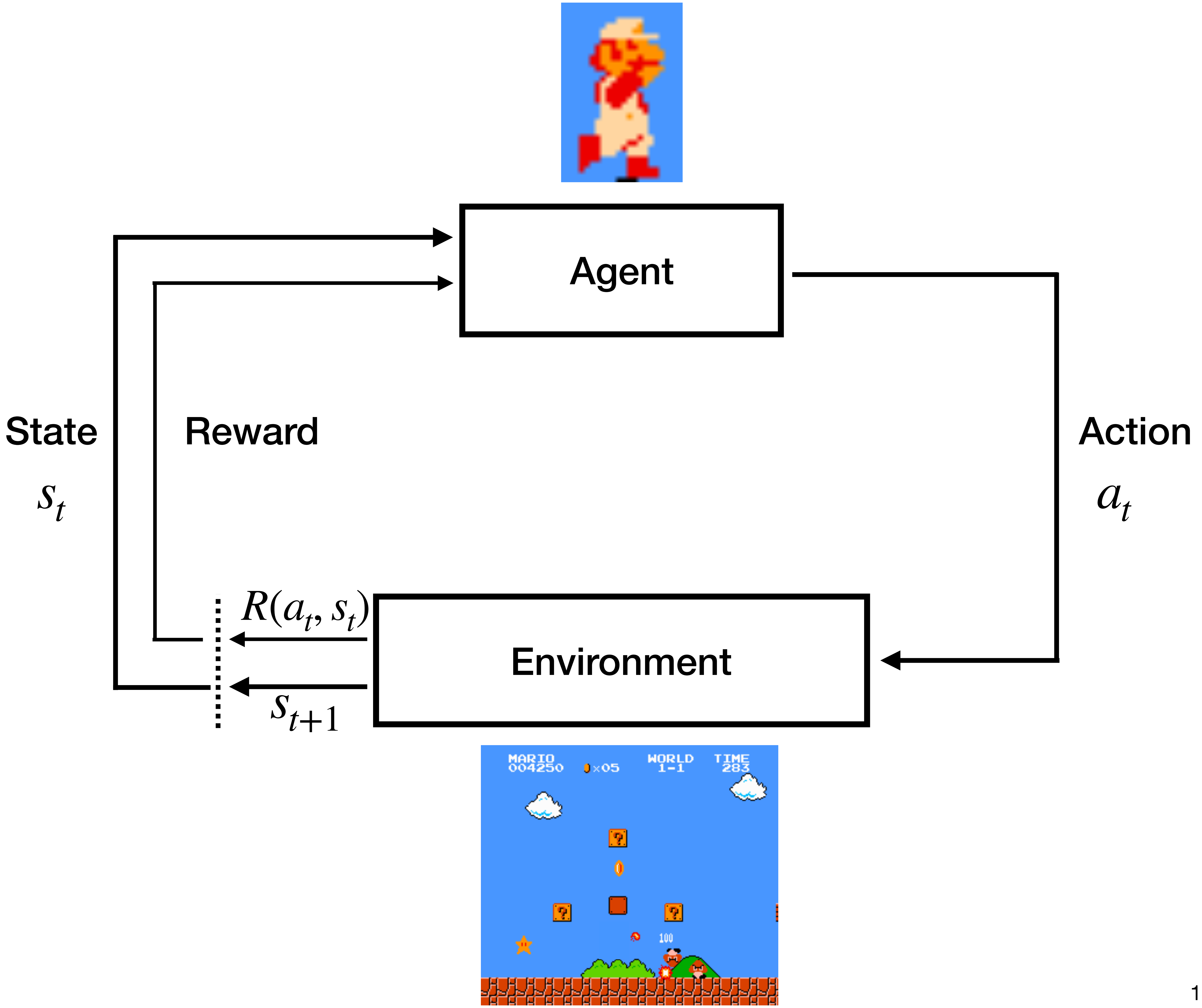
The Agent:



Reinforcement Learning

The Agent:

The Environment:



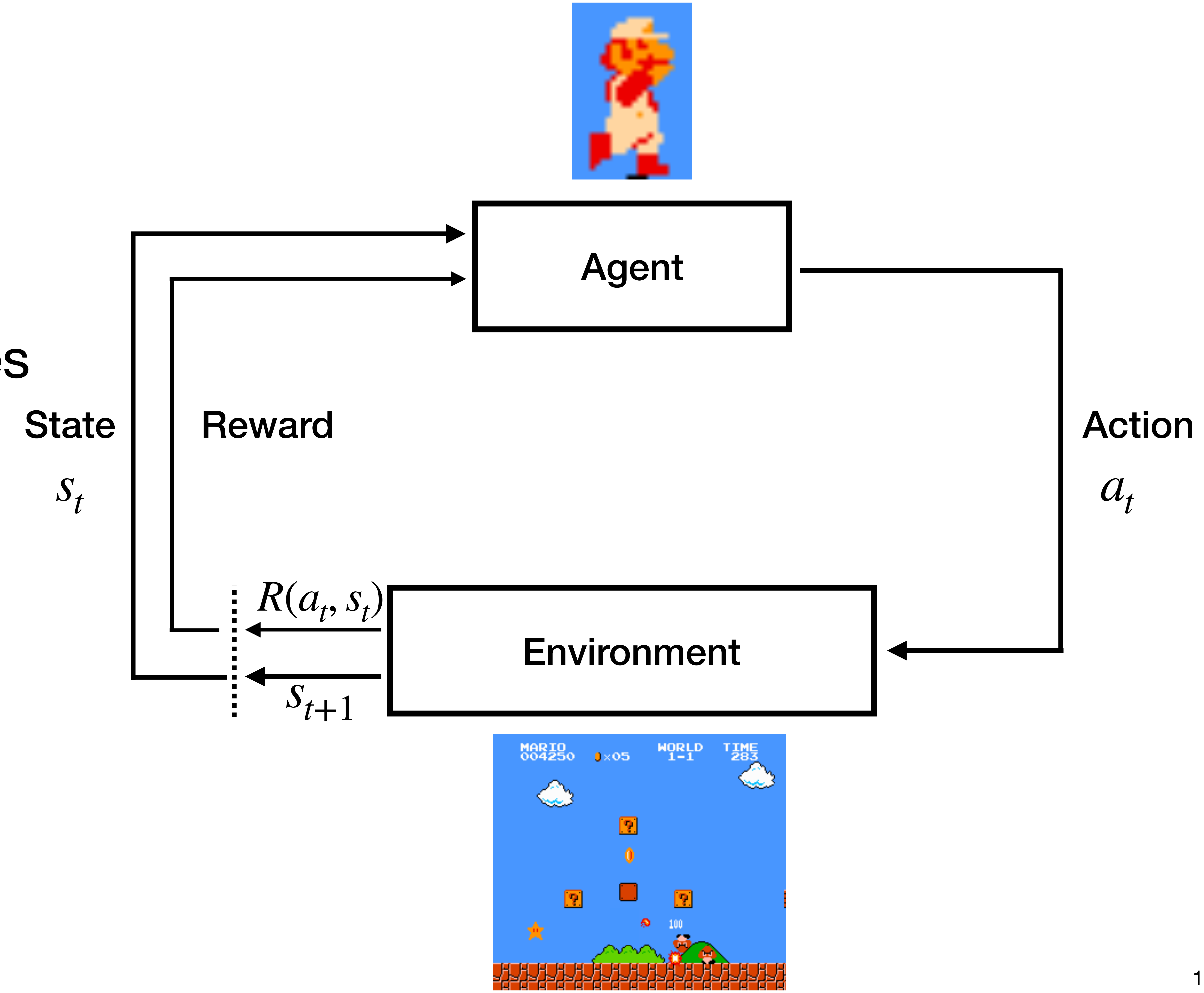
Sutton and Barto (2018 [1998])

Reinforcement Learning

The Agent:

- Selects actions a_t
- Receives feedback from the environment in terms of new states s_{t+1} and rewards $R(a_t, s_t)$

The Environment:



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Reinforcement Learning

The Agent:

- Selects actions a_t
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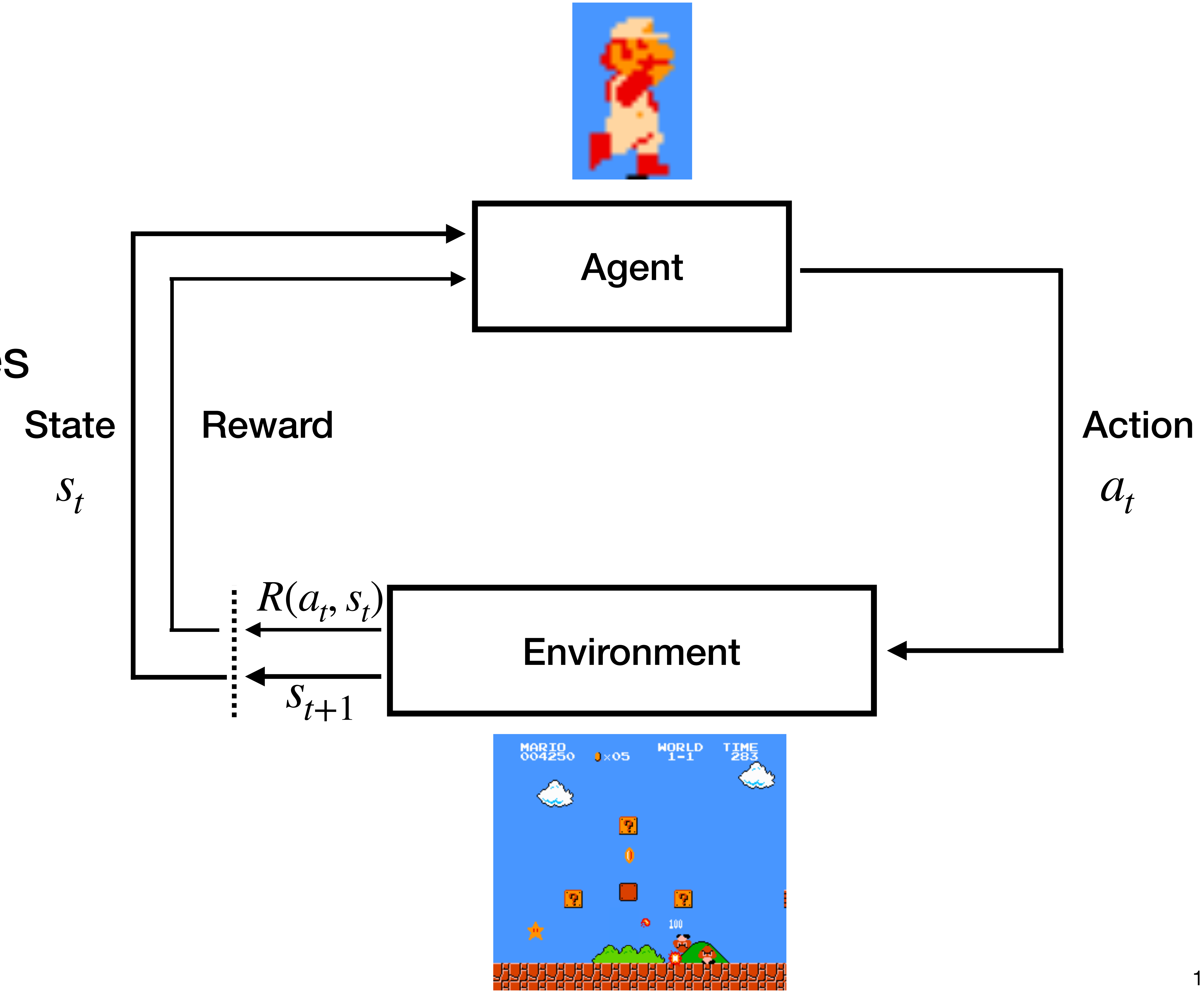
The Environment:

- Governs the transition between states $s_t \rightarrow s_{t+1}$
- Provides rewards $R(a_t, s_t)$



Environment

actions

states

reward

Environment

actions

states

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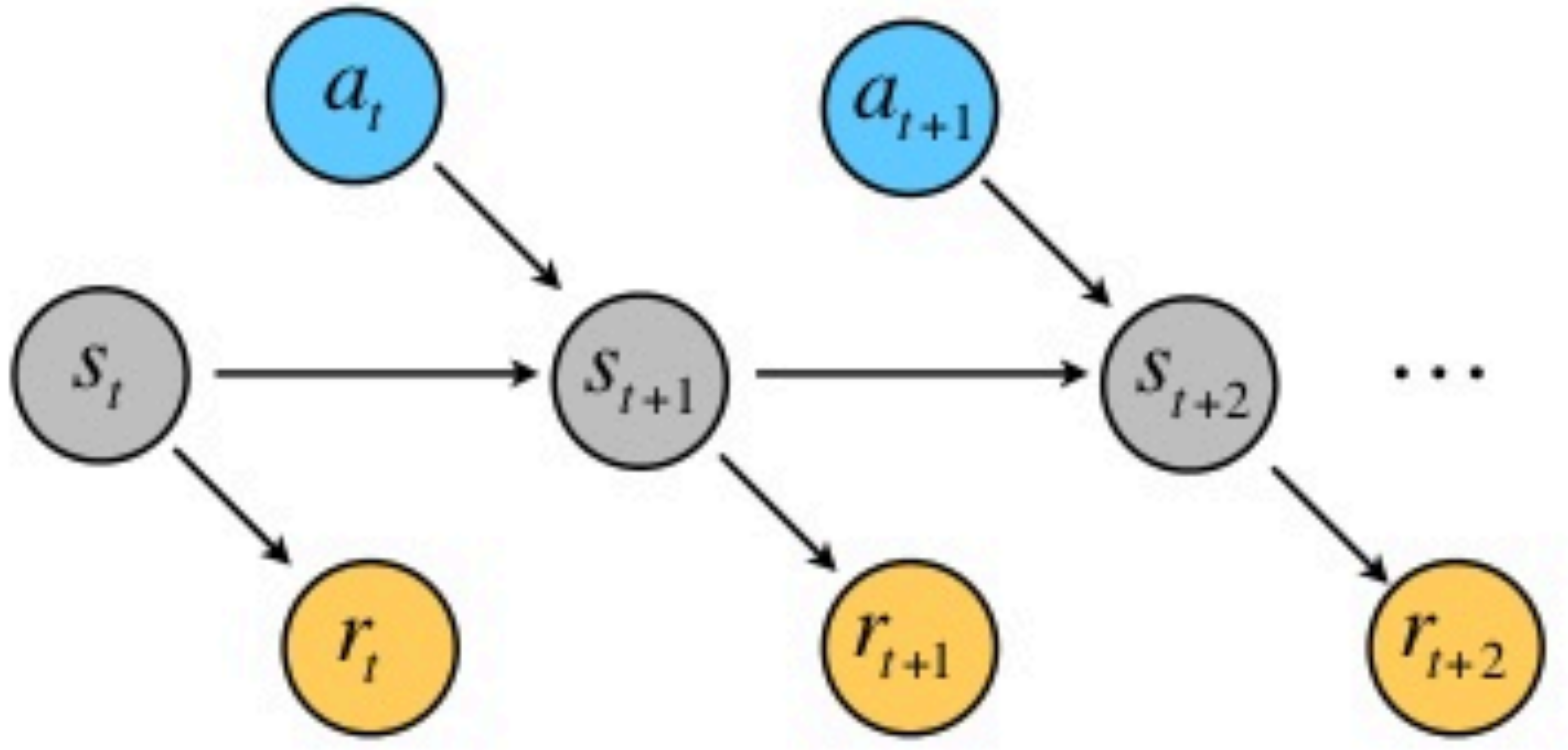


Markov Decision Process (MDP)

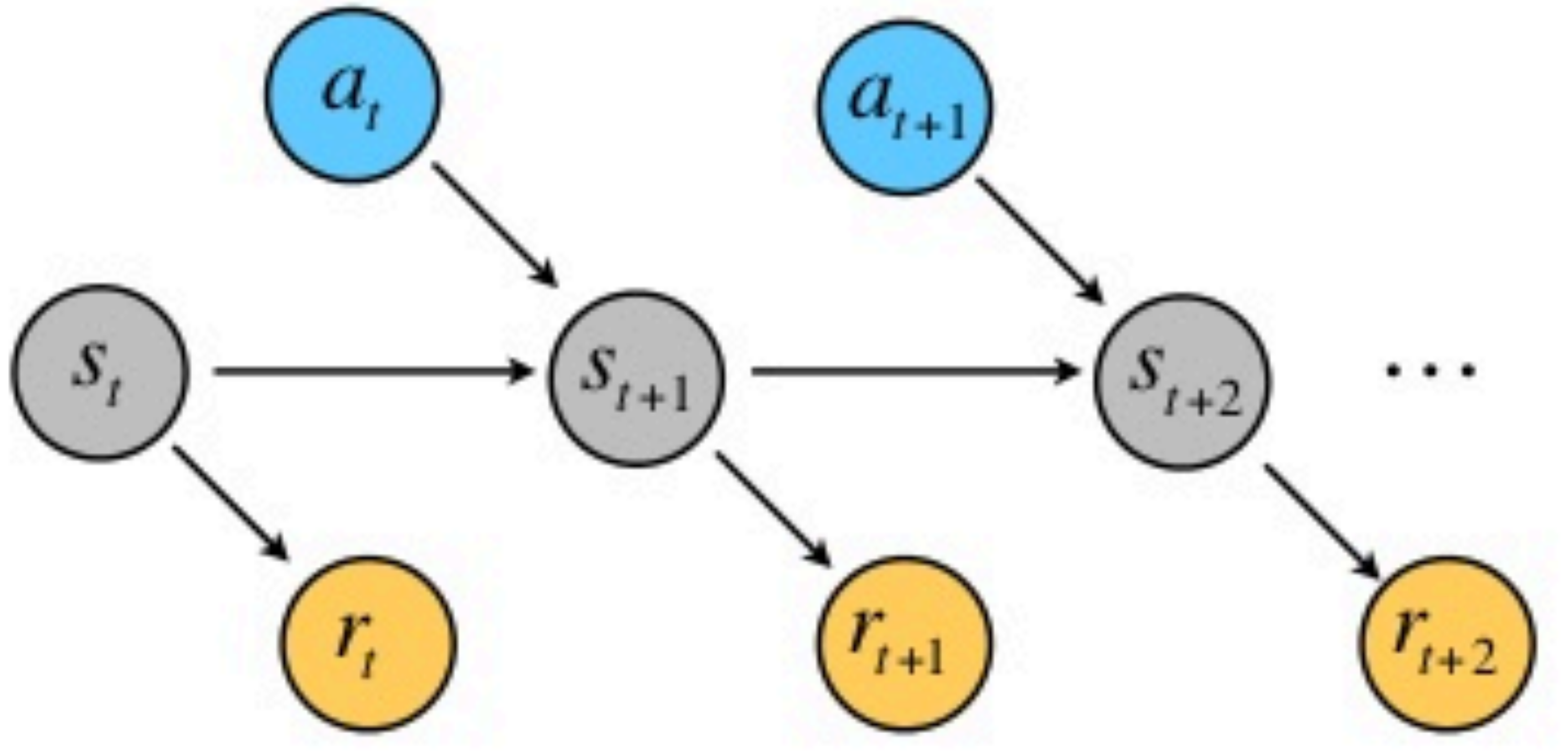
- Markov Principle: simplifying assumption that the system is fully defined by only the previous state $P(s_{t+1} \mid s_t, a_t)$

Environment

actions

states

reward



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What are states?

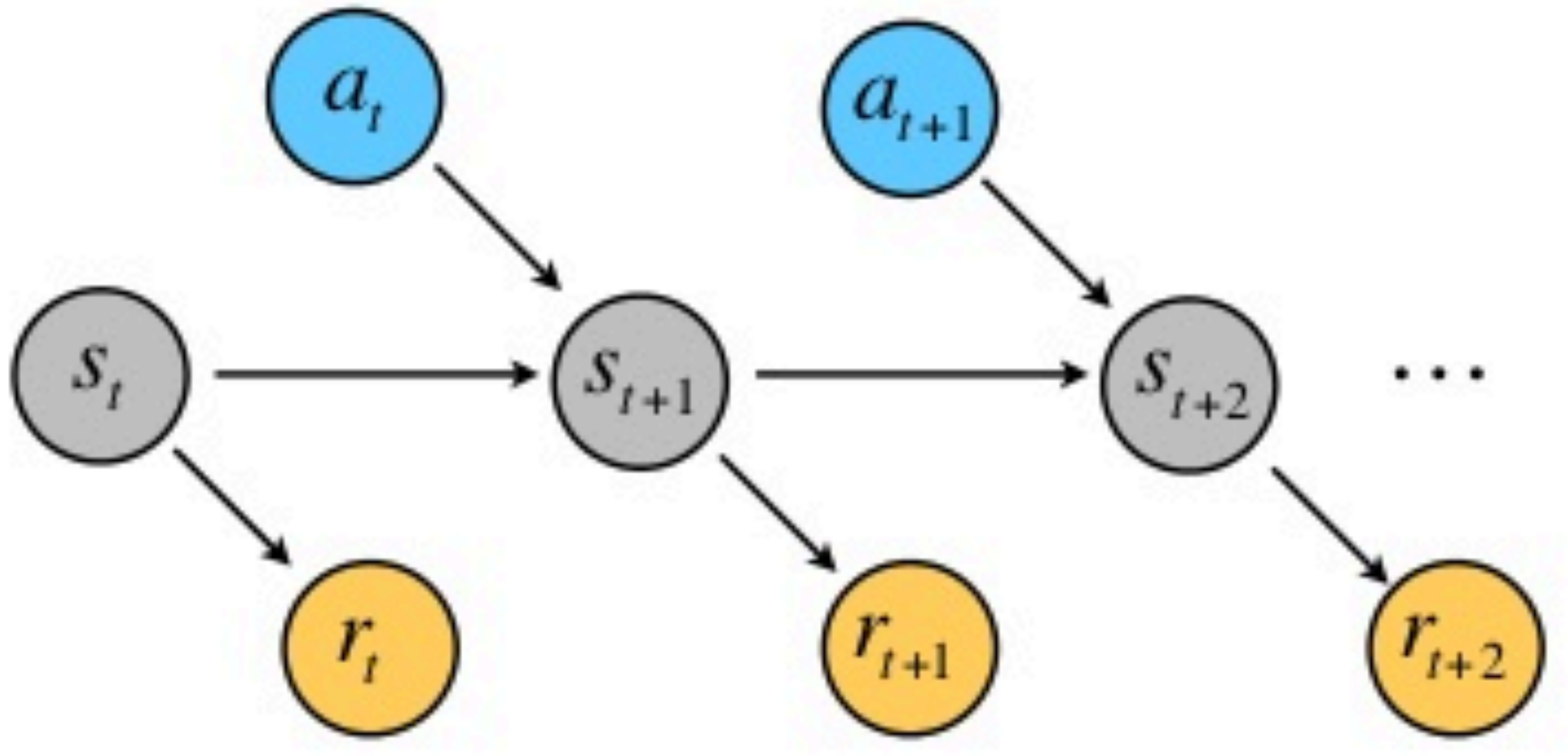
- Locations on a grid, pixels on a screen, feature values, etc…

Environment

actions

states

reward



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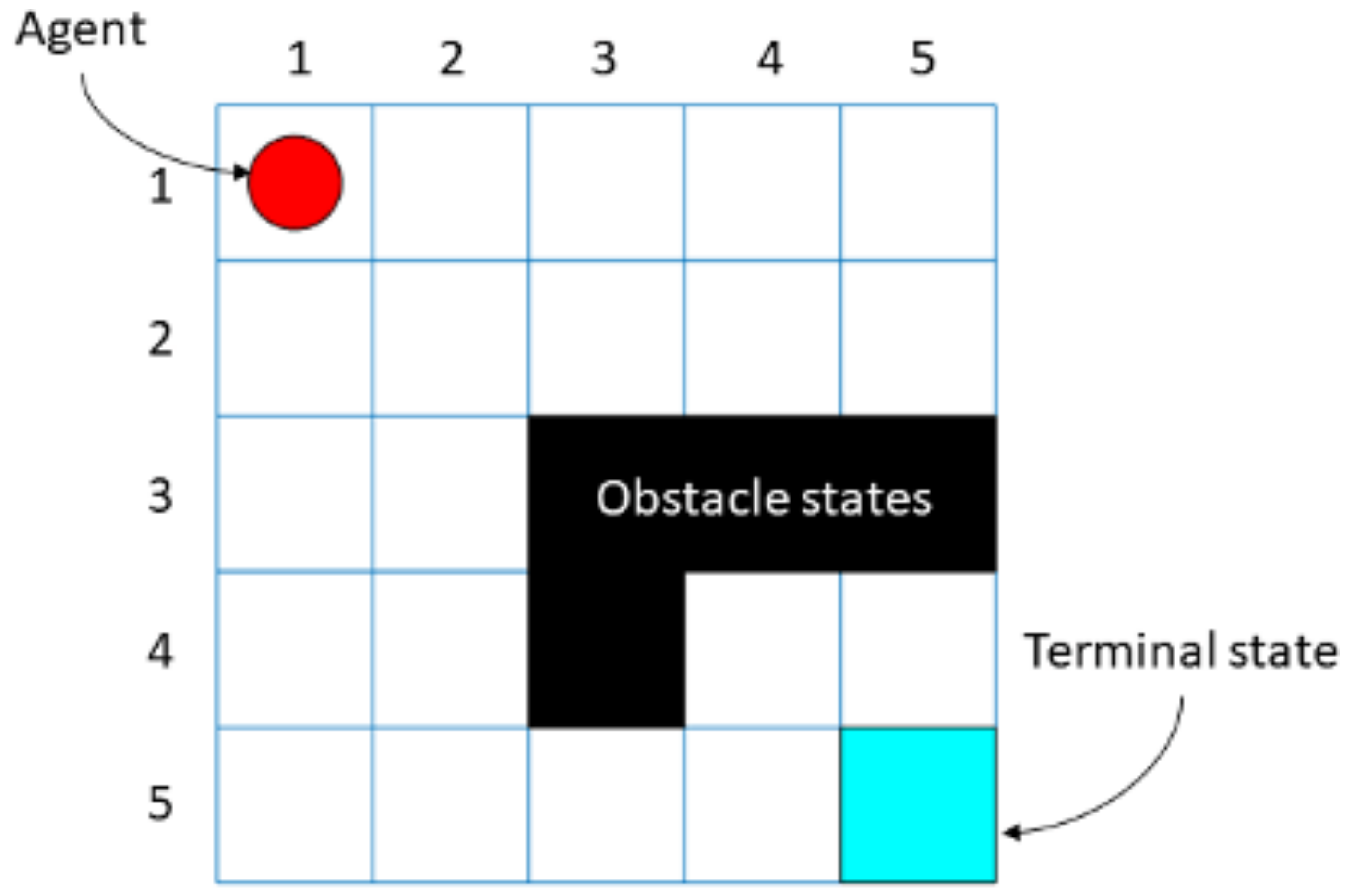


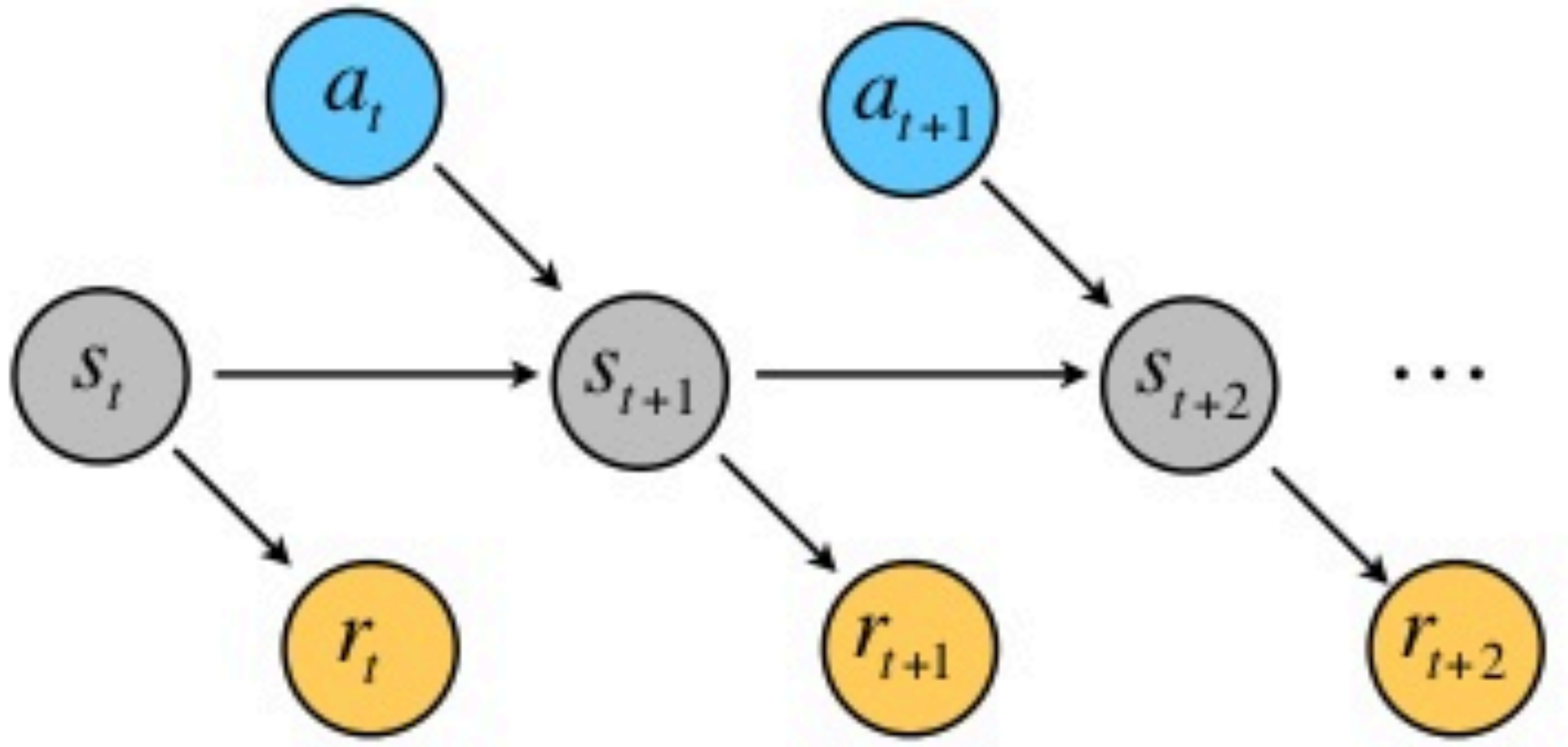
Grid World

Environment

actions

states

reward



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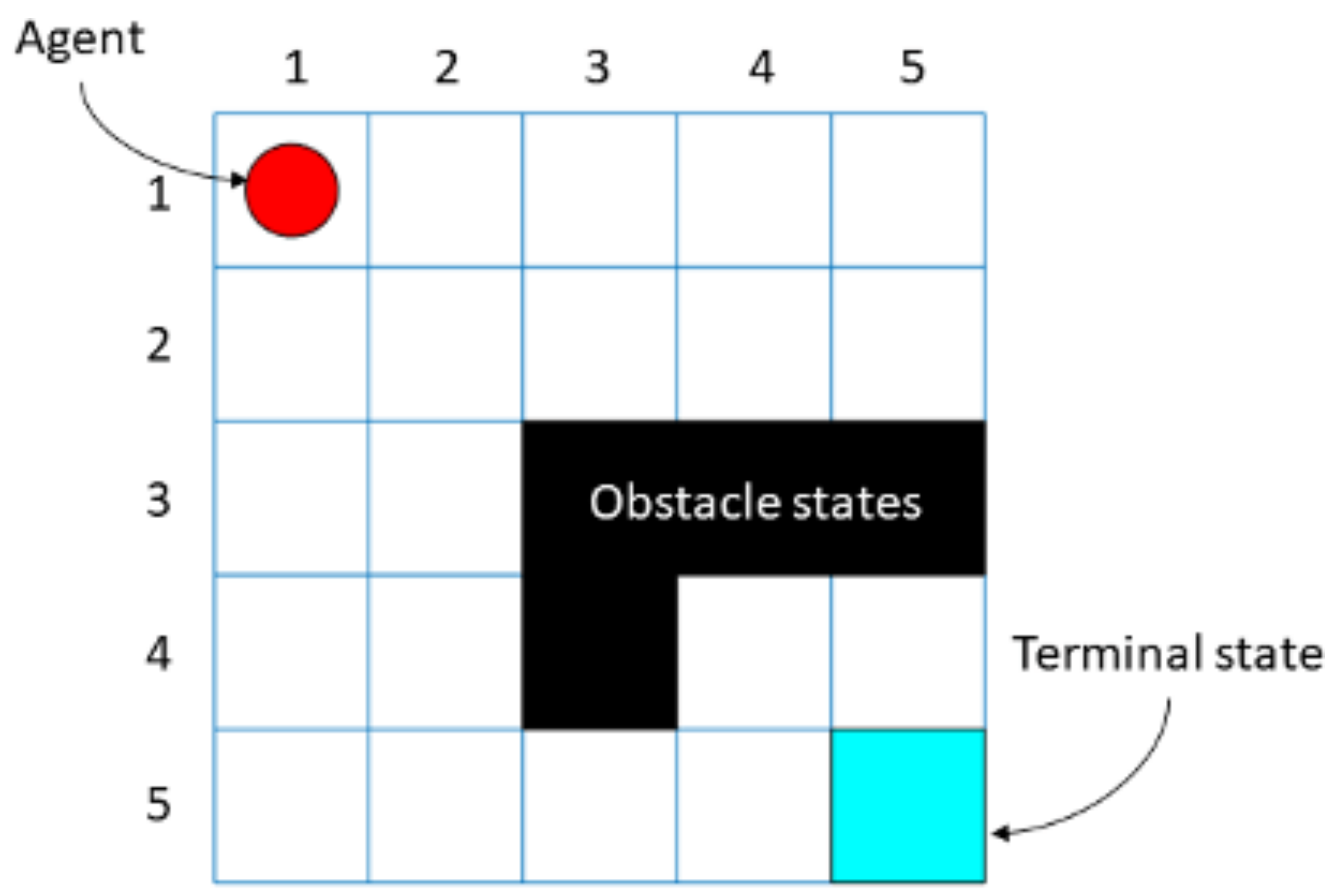

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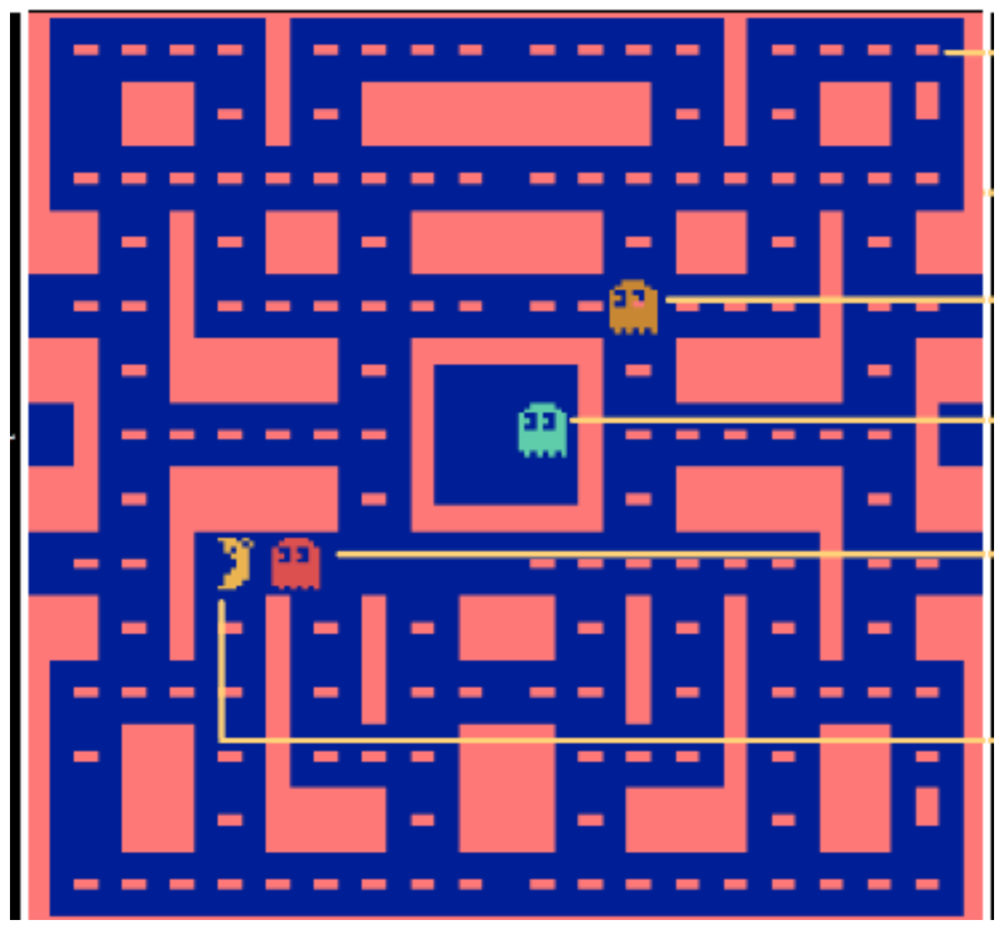
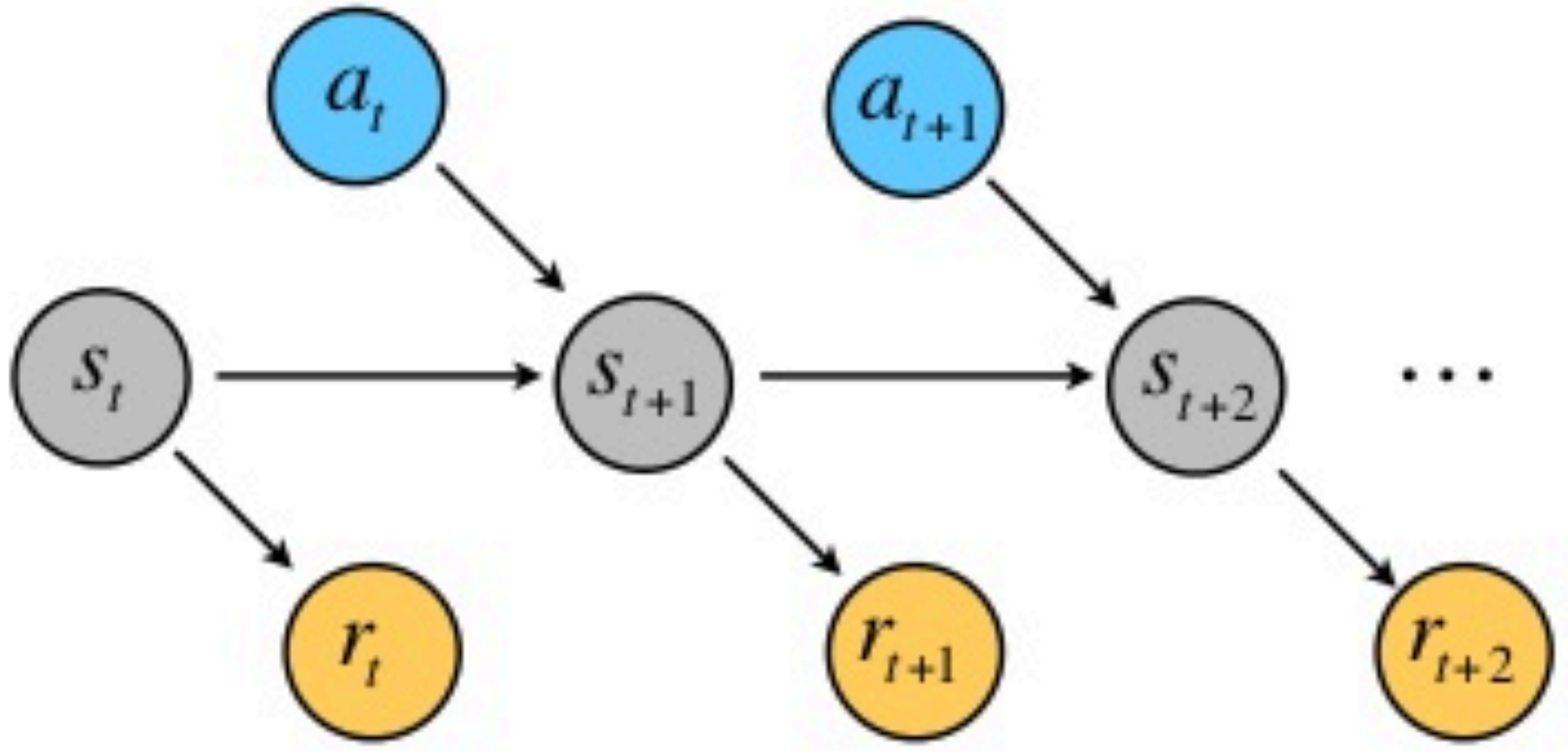


Environment

actions

states

reward



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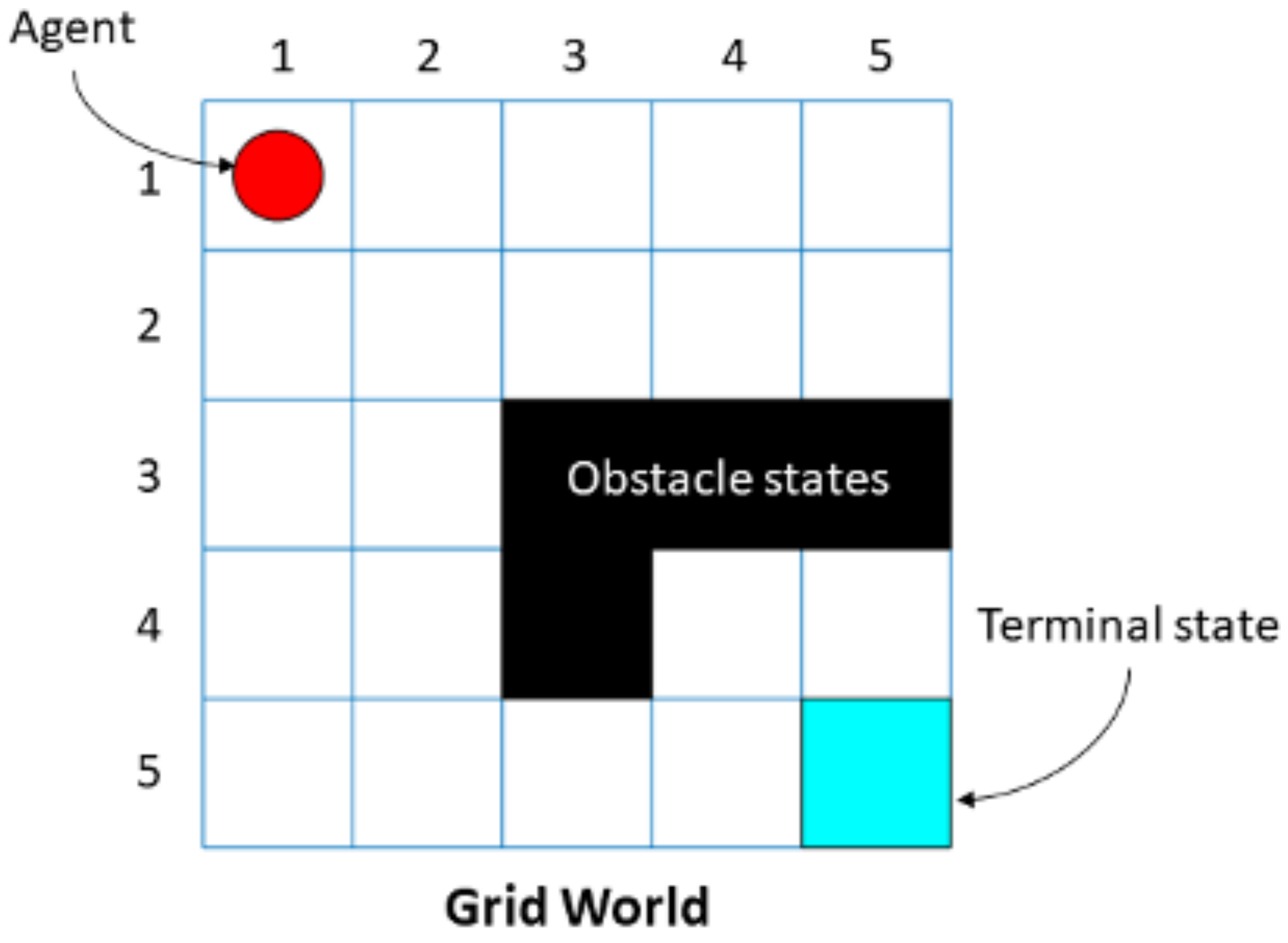
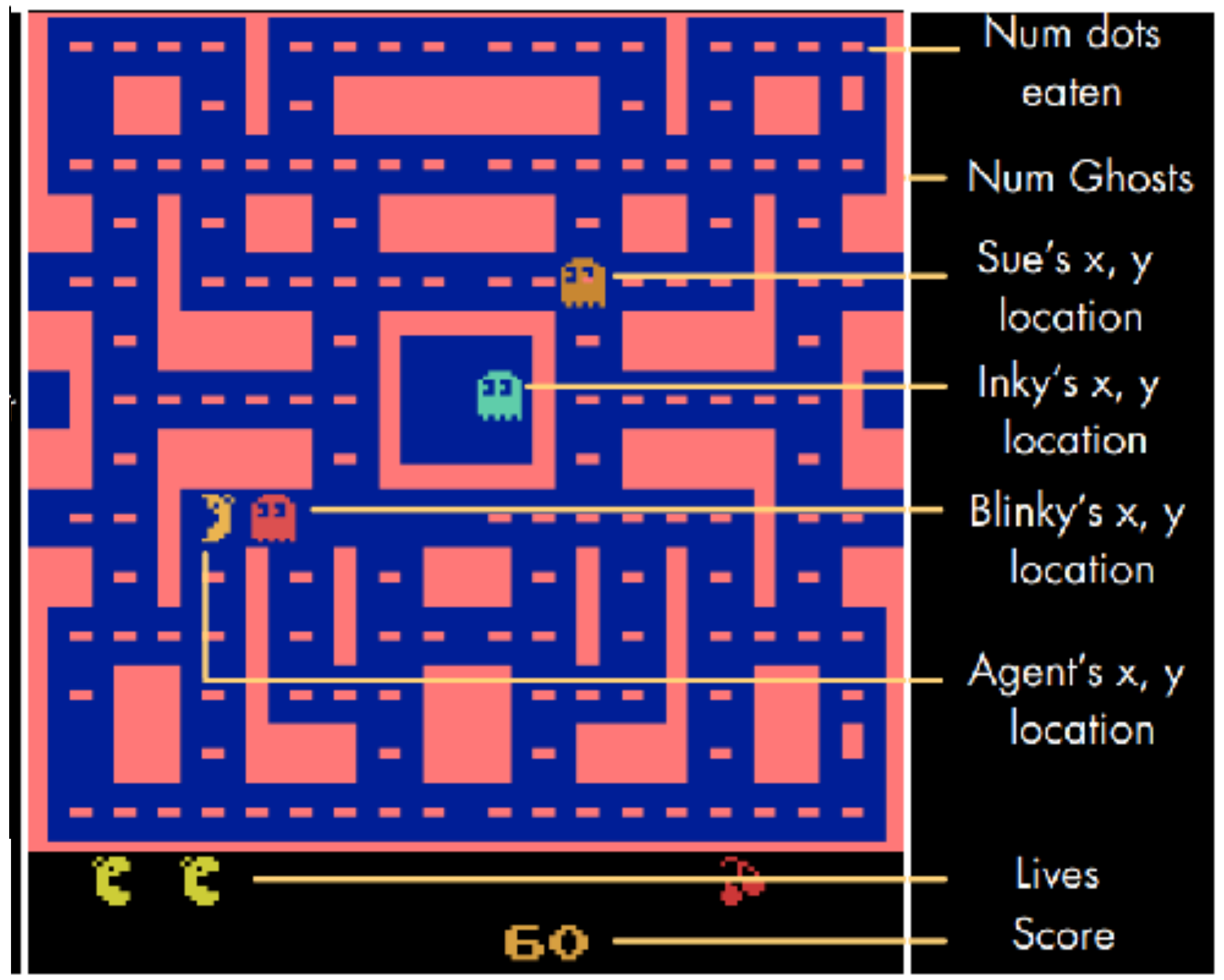


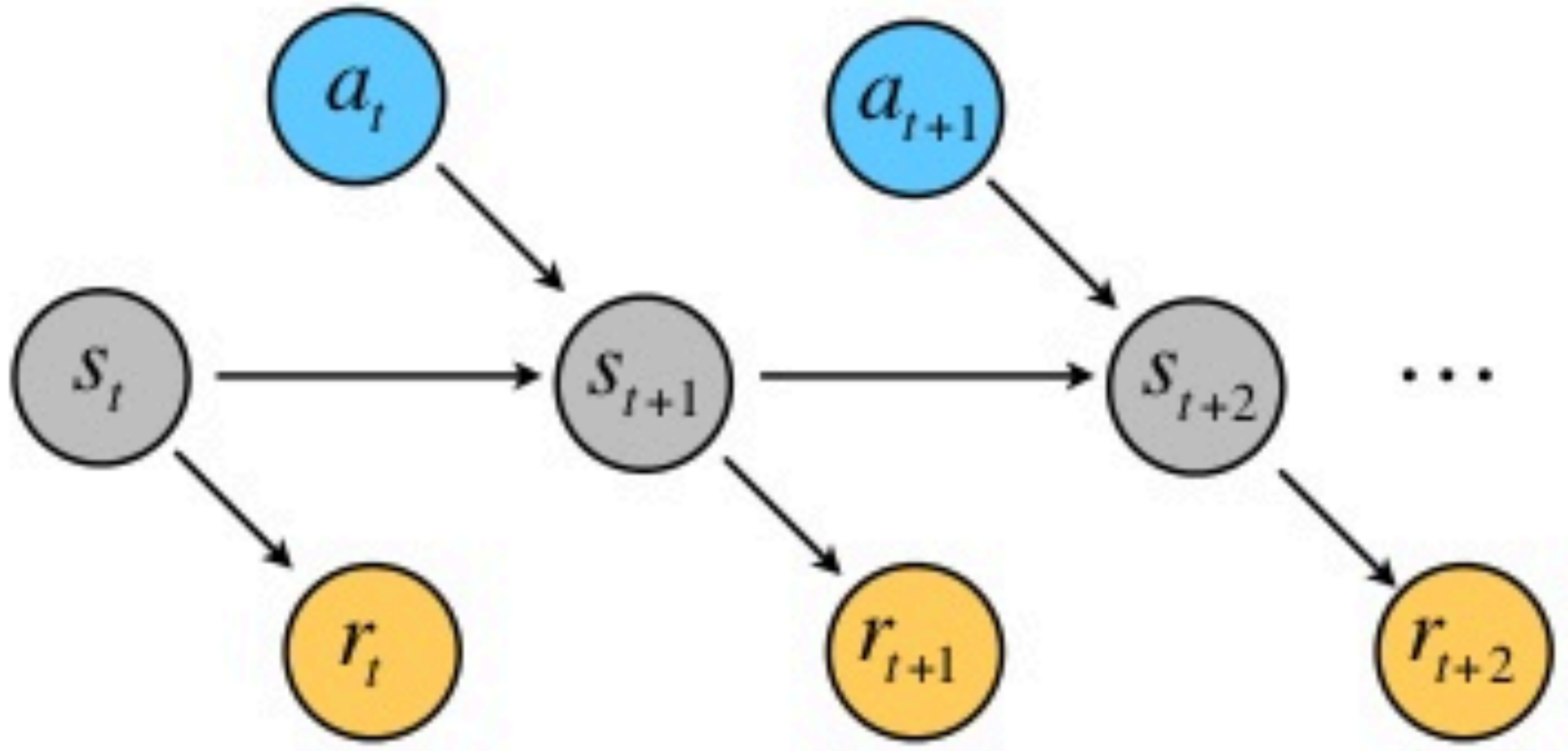


Environment

***a*ctions**

***s*tates**

***r*eward**



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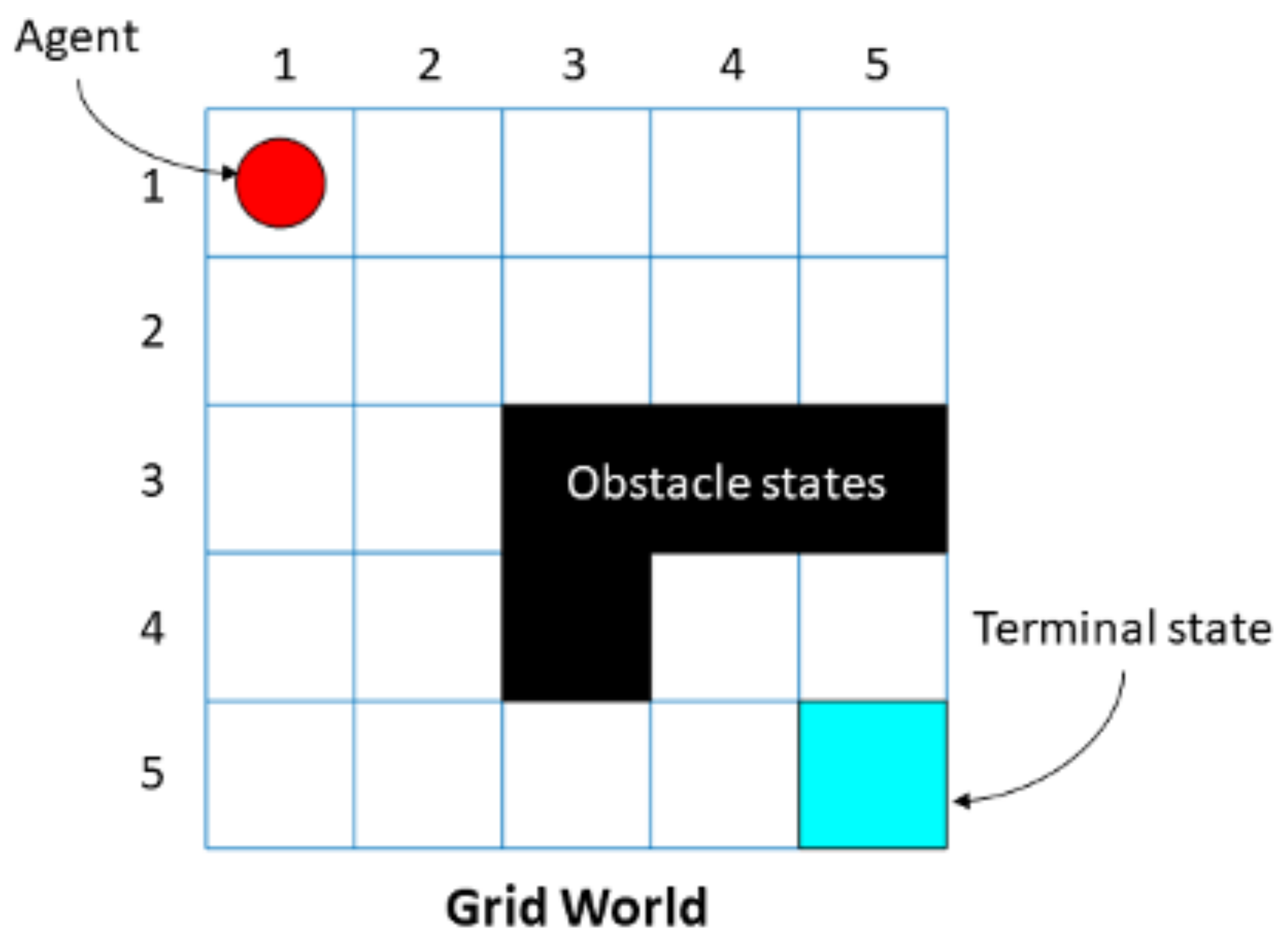
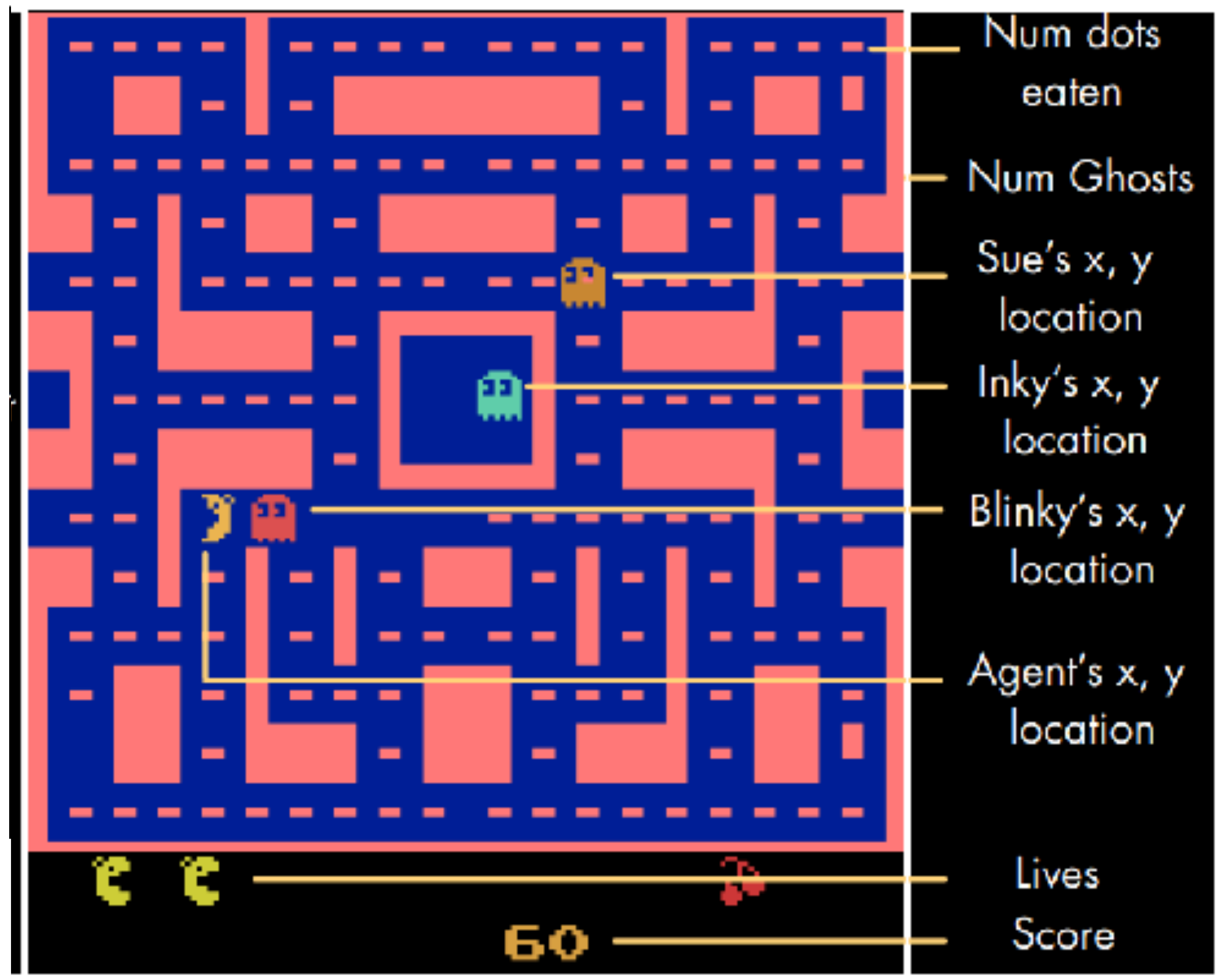
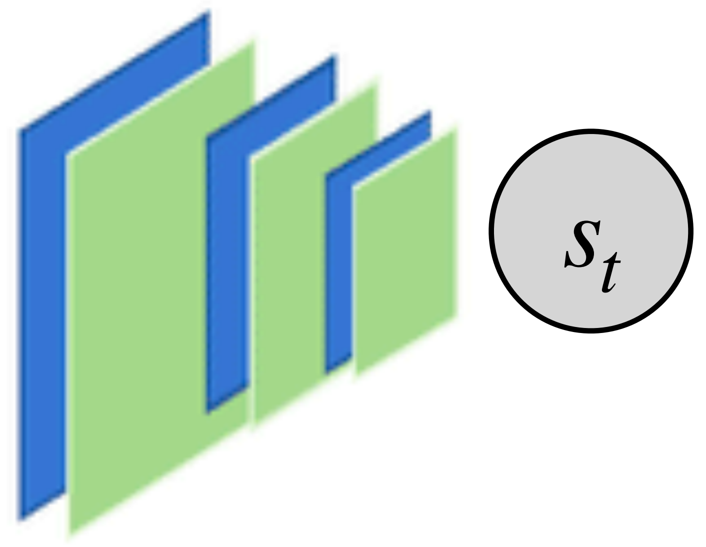






Agent

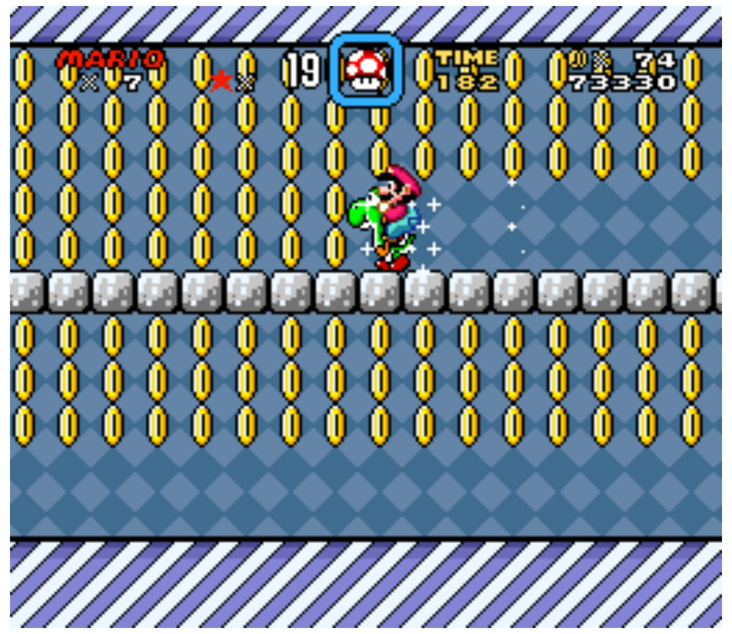
Agent

- **Represent Past Experiences**

- **Implement a Policy that Maximizes Reward**

Agent

- **Represent Past Experiences**
  - How good is a state? $V(s_t)$



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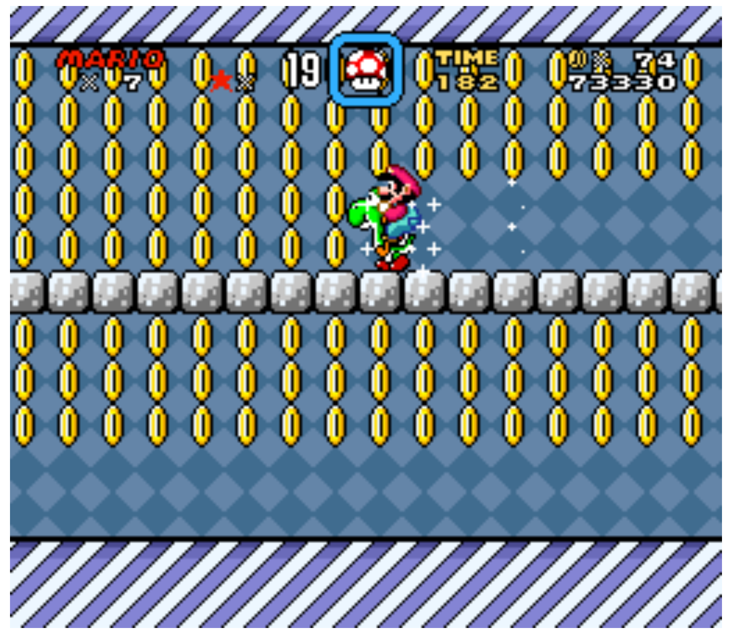

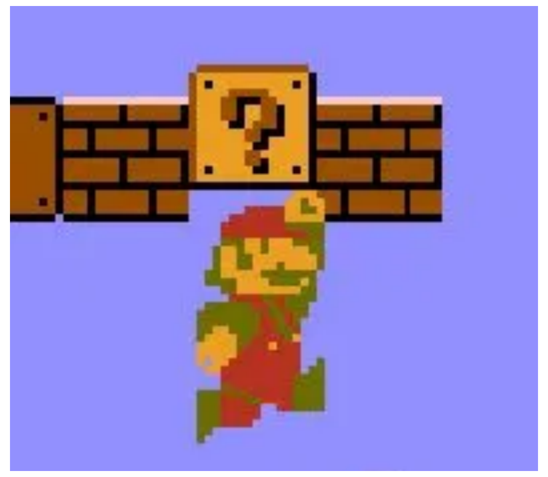

- **Implement a Policy that Maximizes Reward**

Agent

- **Represent Past Experiences**
    - How good is a state? $V(s_t)$
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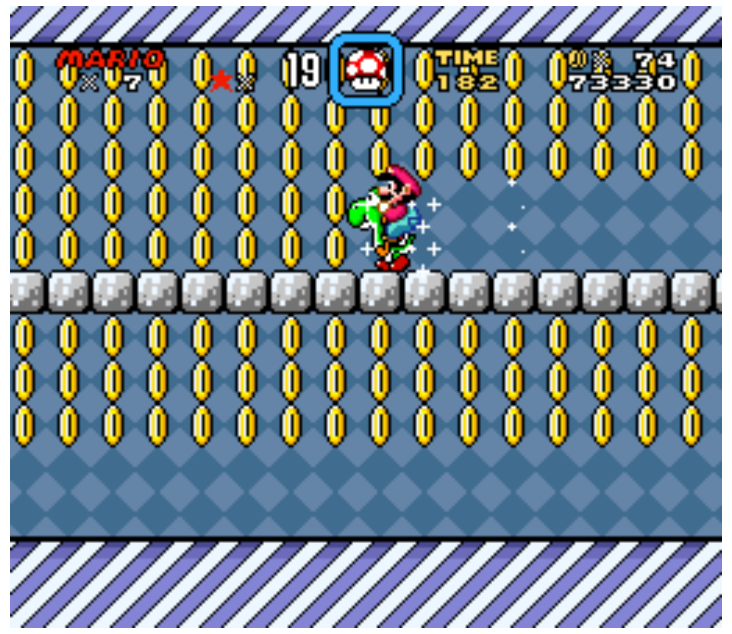
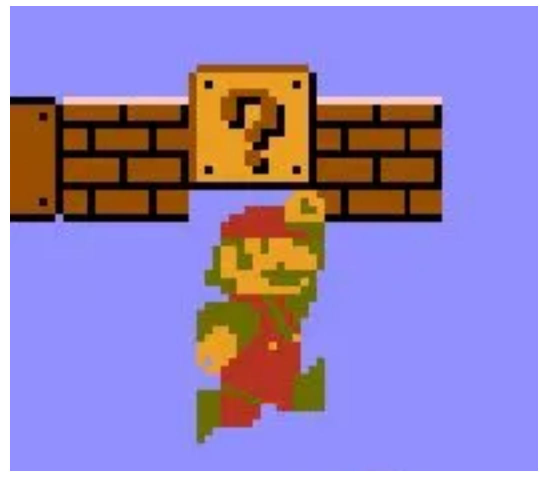
    - How good is a *trajectory* $\tau = (s_0, a_0, s_1, a_1, \ldots)$?
- **Implement a Policy that Maximizes Reward**





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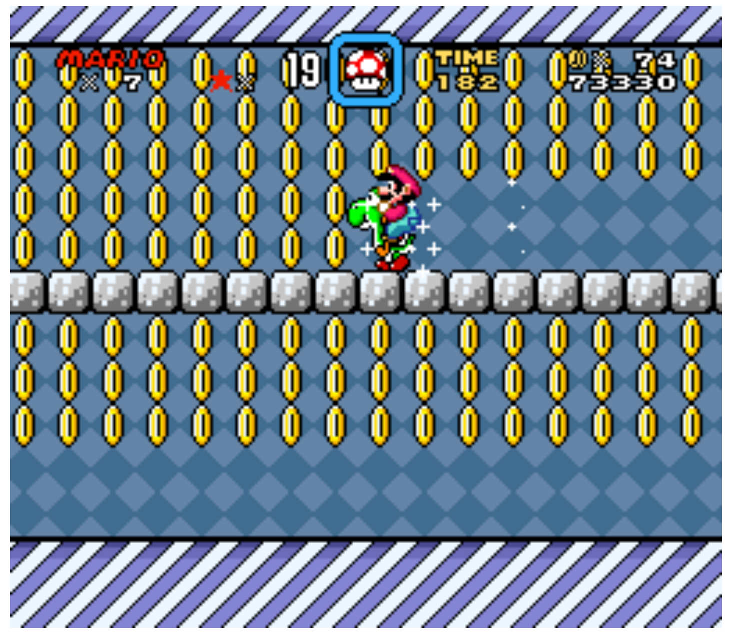
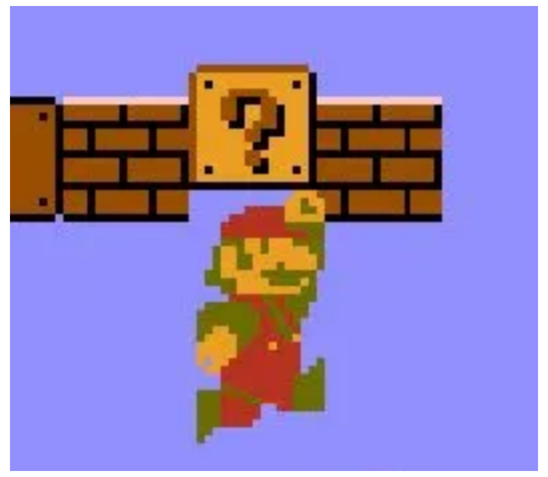
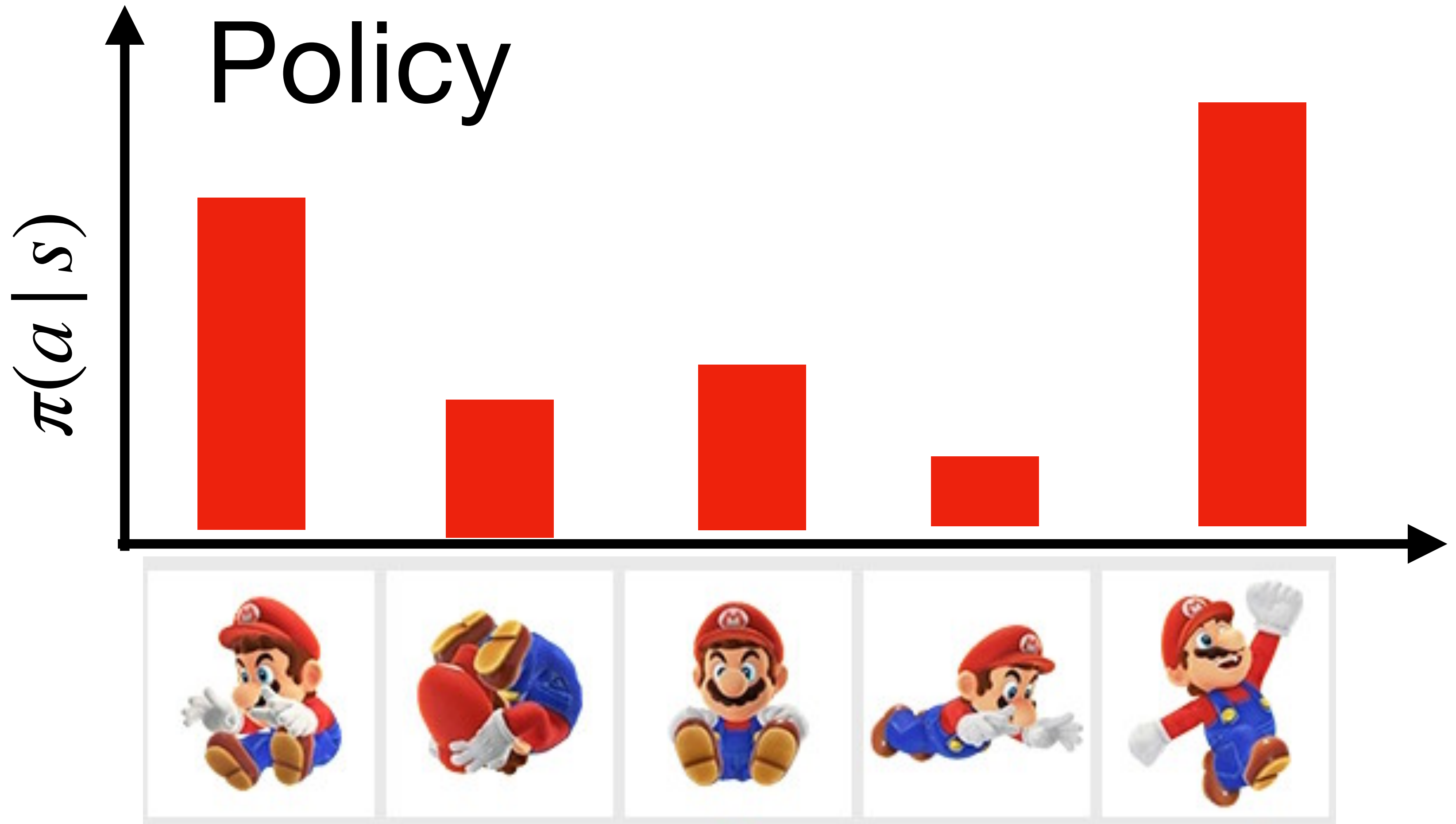




- **Implement a Policy that Maximizes Reward**
    - π defines how to act, where $\pi(a \mid s)$ is the probability of selecting action a in state s
    - sample actions from the policy $a_t \sim \pi$



Normative vs. Descriptive

RL as a **normative** framework:

- How *should* a rational agent behave when learning from the environment?
- Which learning mechanisms and which policies lead to better outcomes?

RL as a **descriptive** framework:

- How *does* an agent update beliefs and select actions when learning from the environment?
- Which learning mechanisms and which policies provide better descriptions of behavior

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Normative vs. Descriptive

RL as a **normative** framework:

- How *should* a rational agent behave when learning from the environment?
- Which learning mechanisms and which policies lead to better outcomes?

RL as a **descriptive** framework:

- How *does* an agent update beliefs and select actions when learning from the environment?
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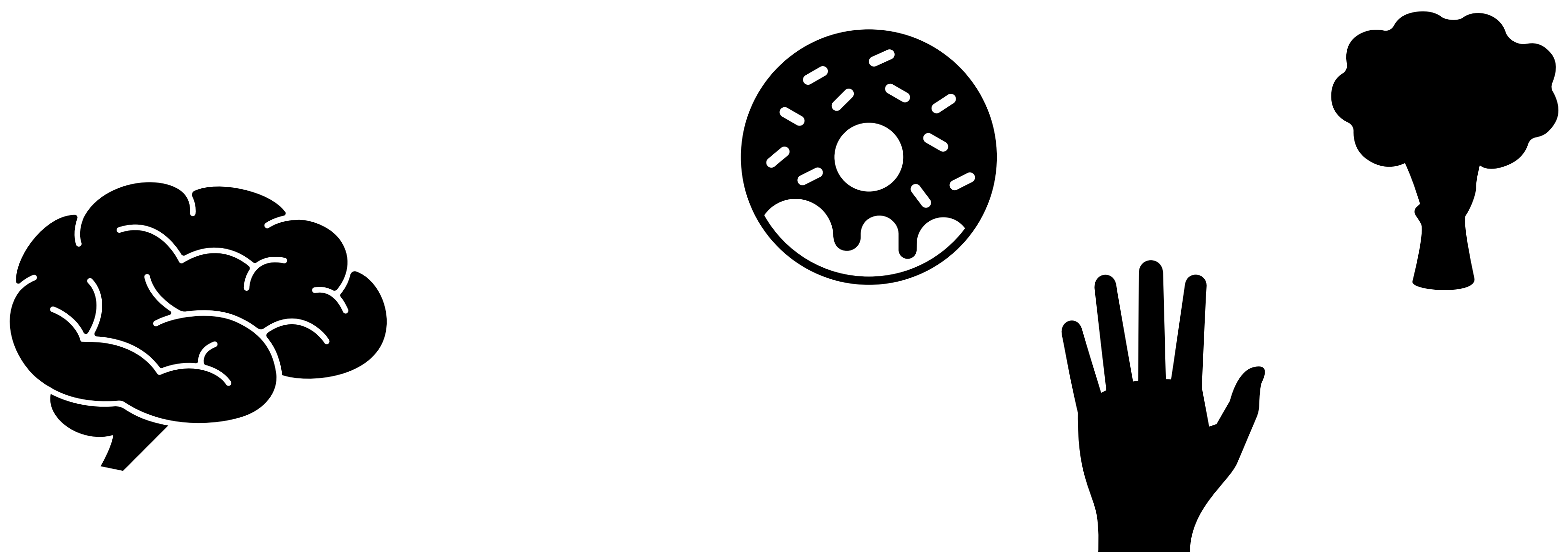
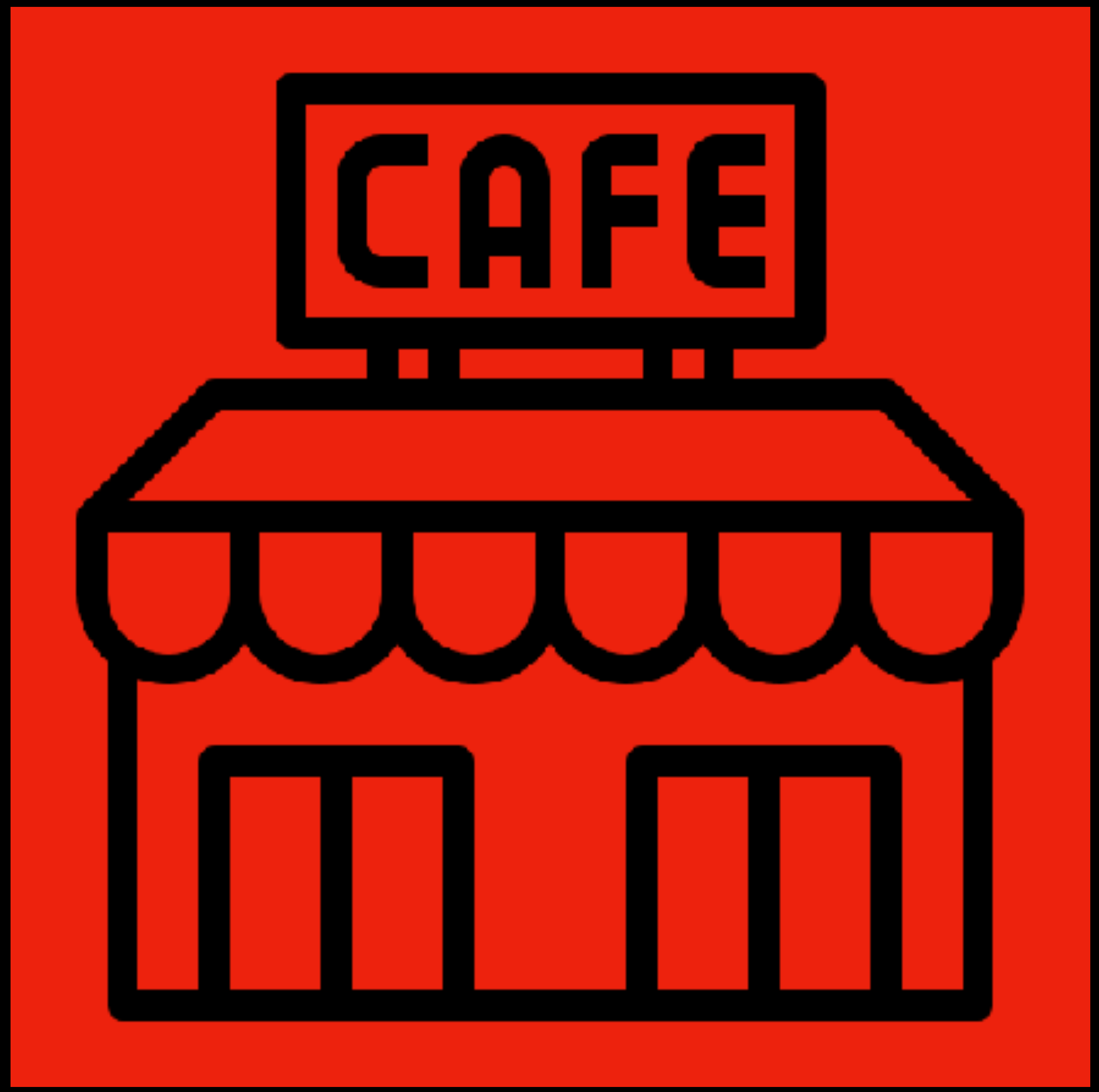


Simplest RL problem & simplest RL model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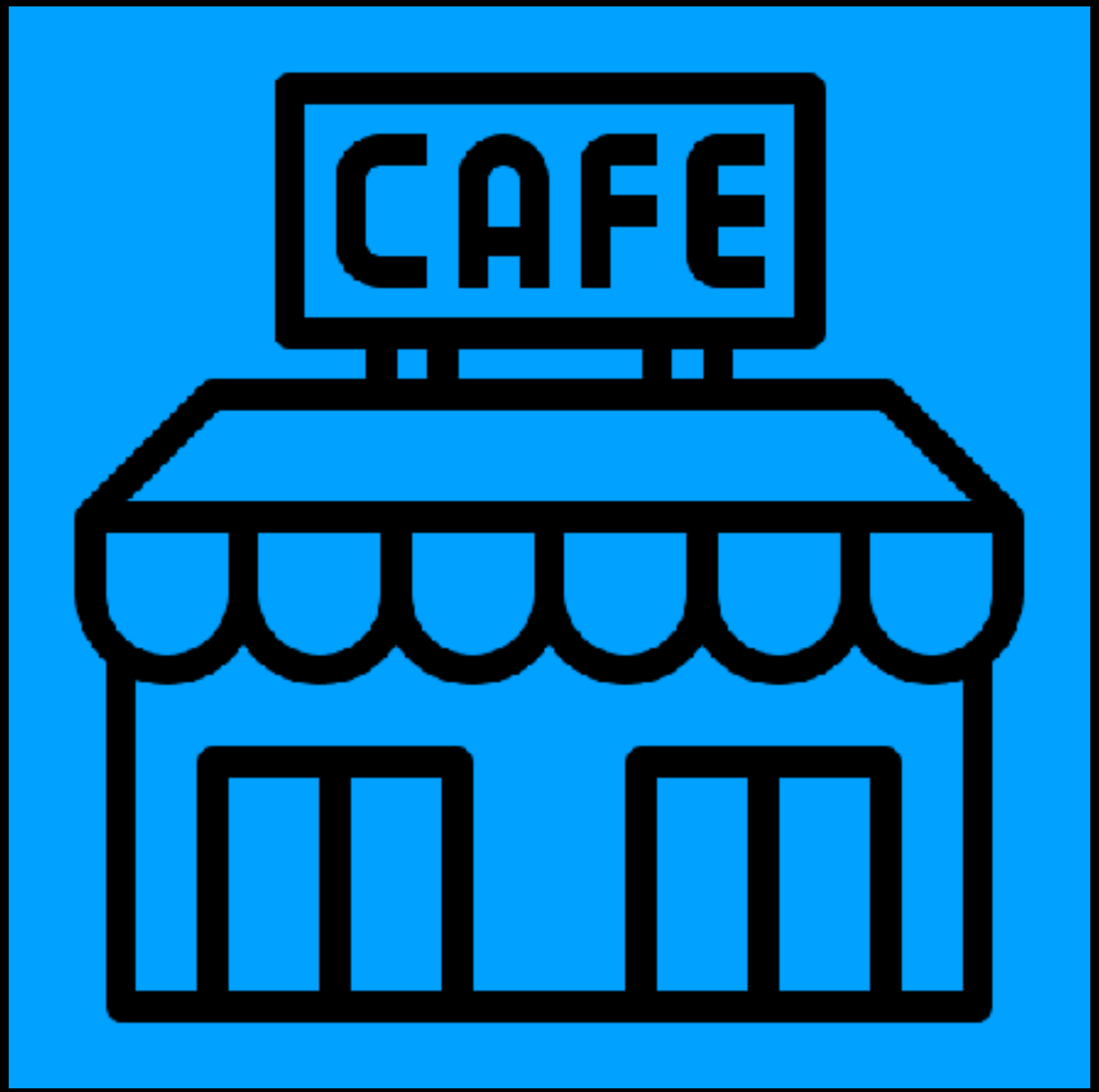
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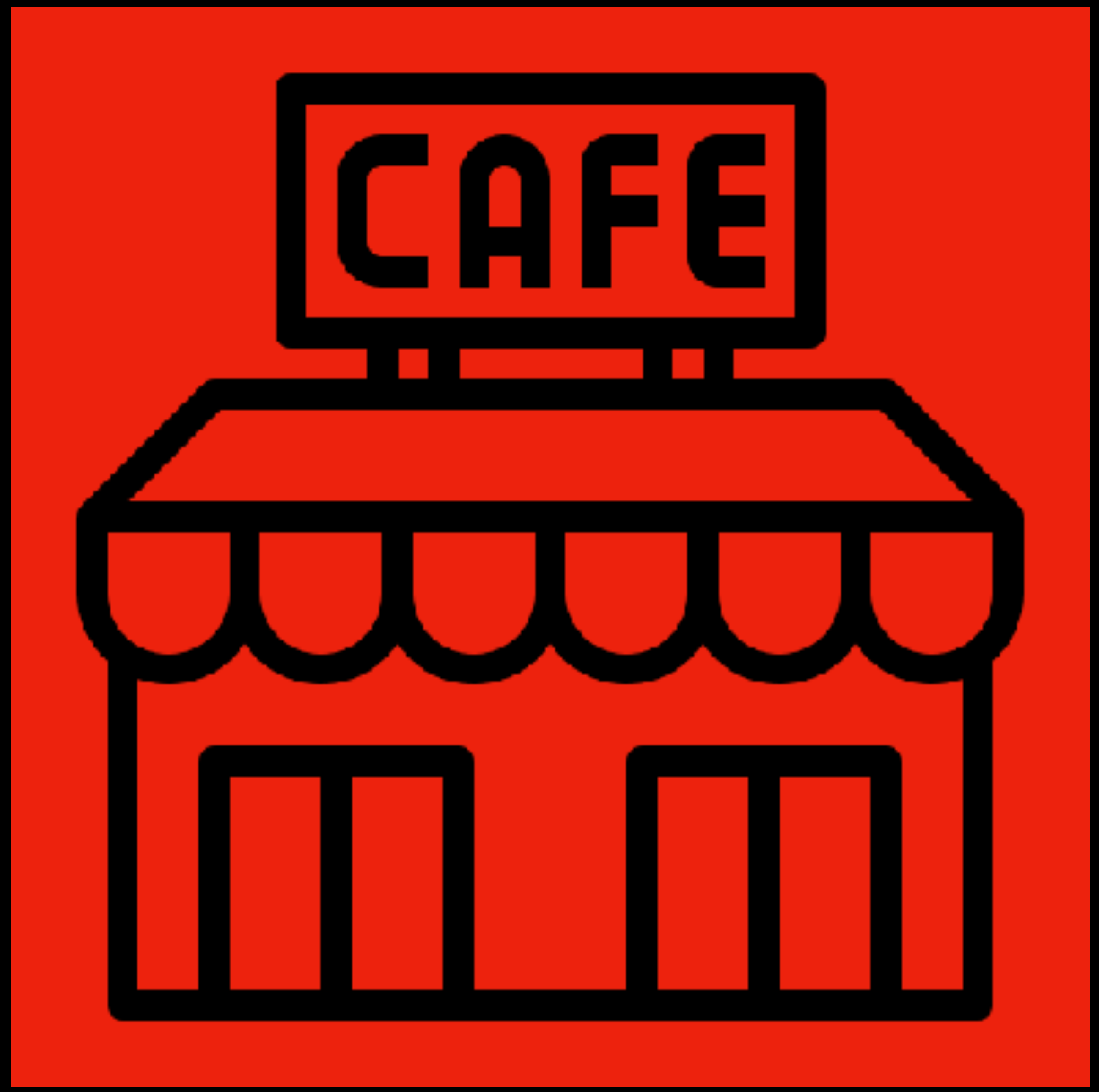
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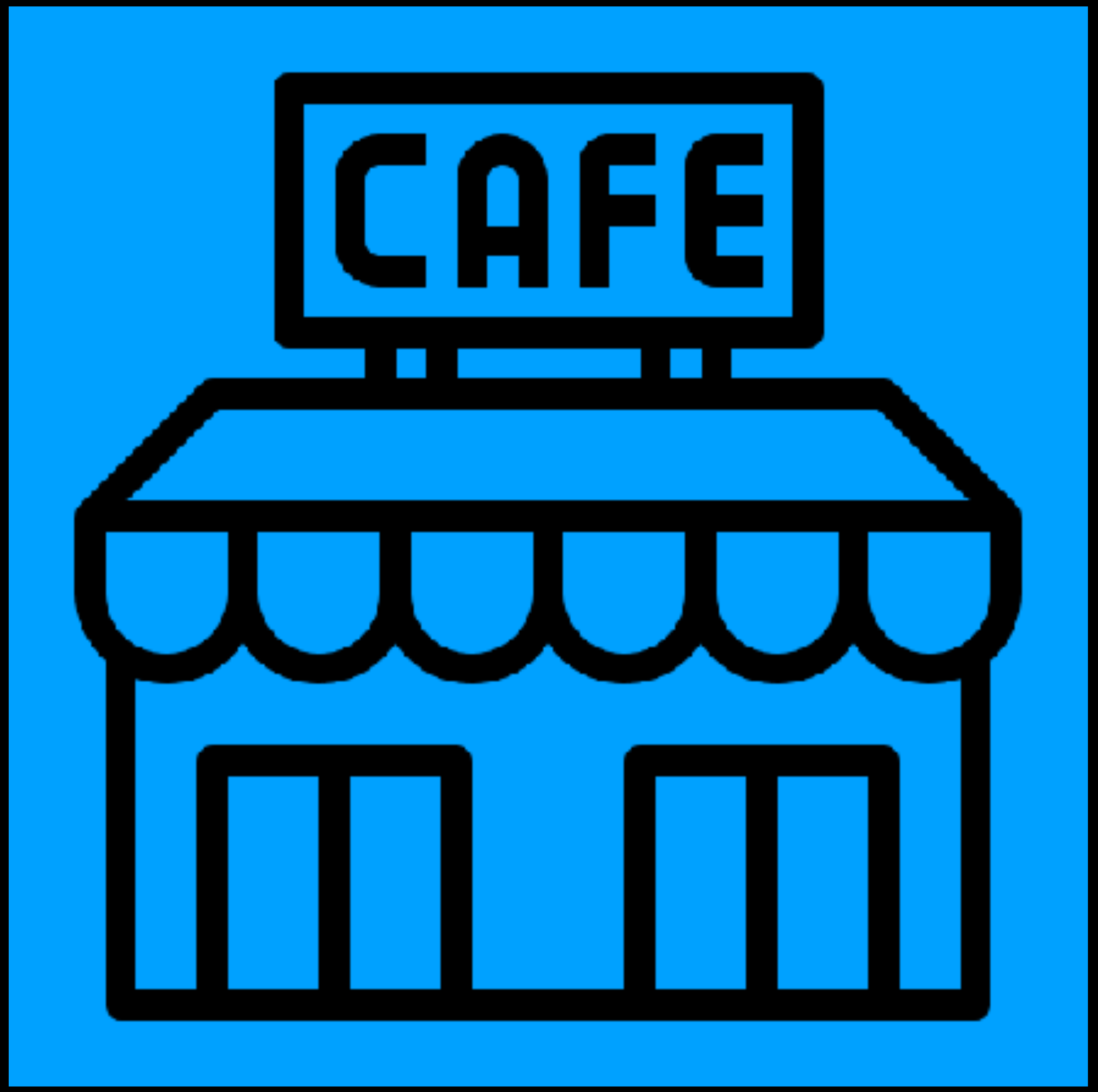
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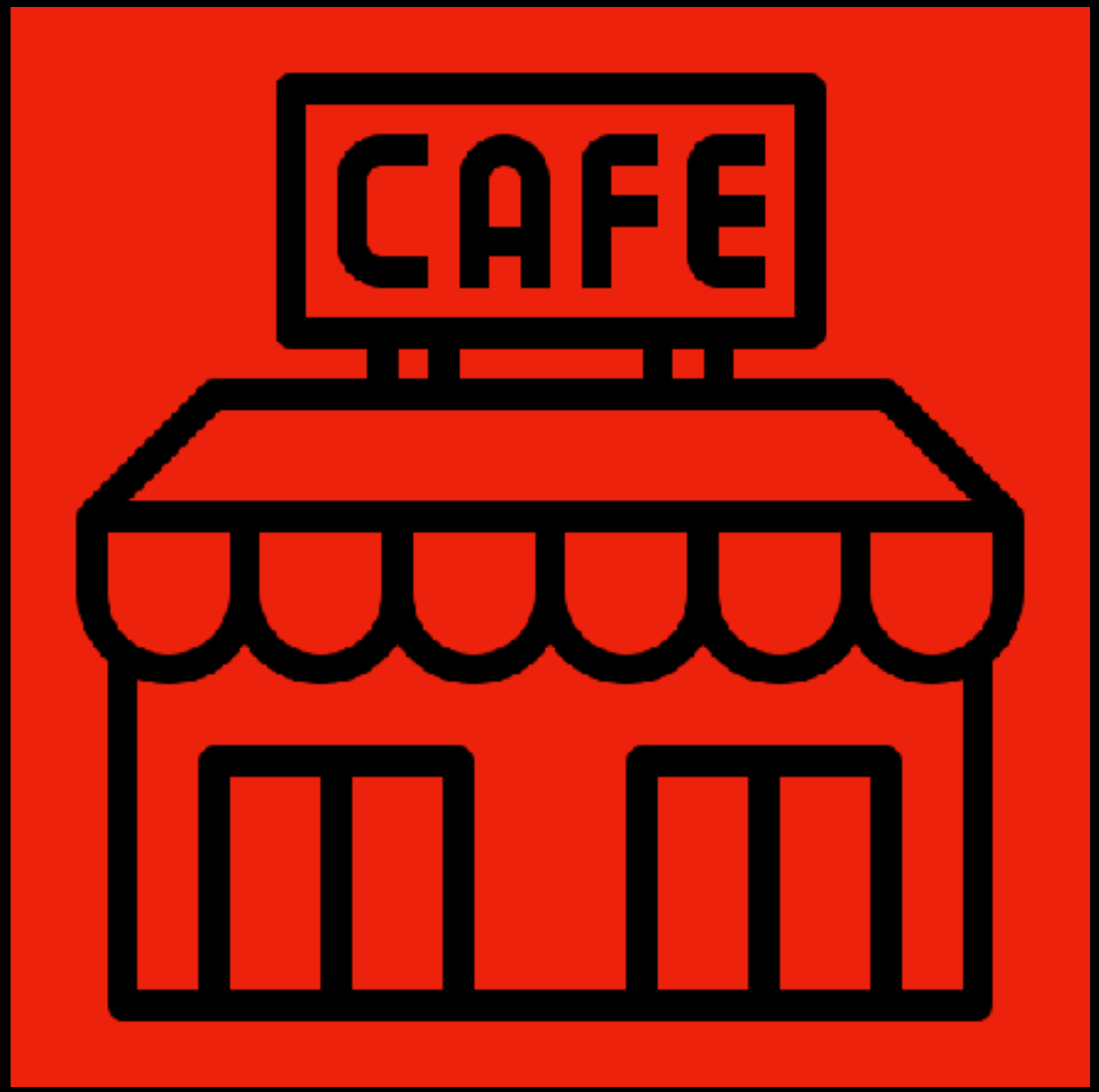
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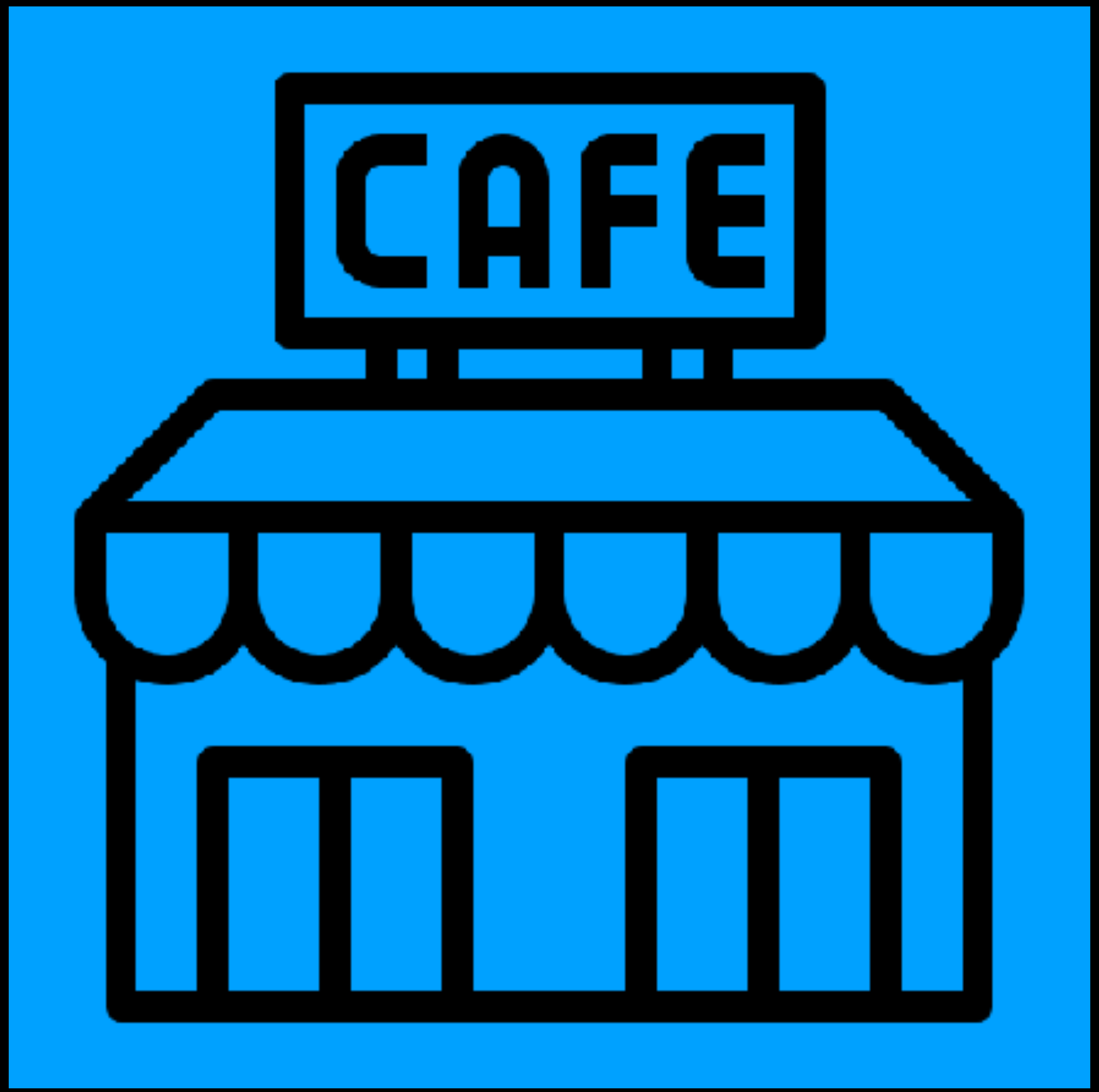
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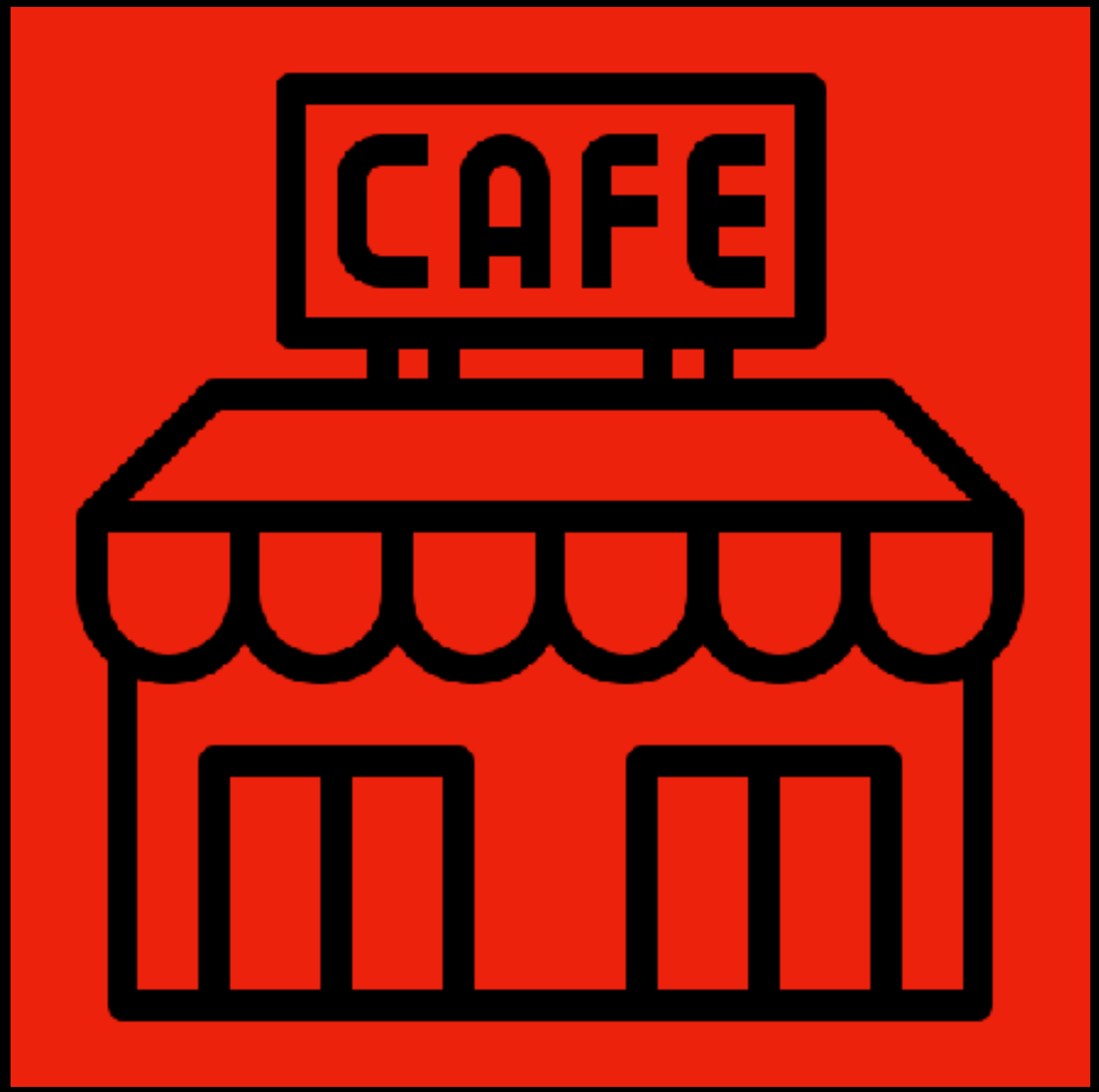
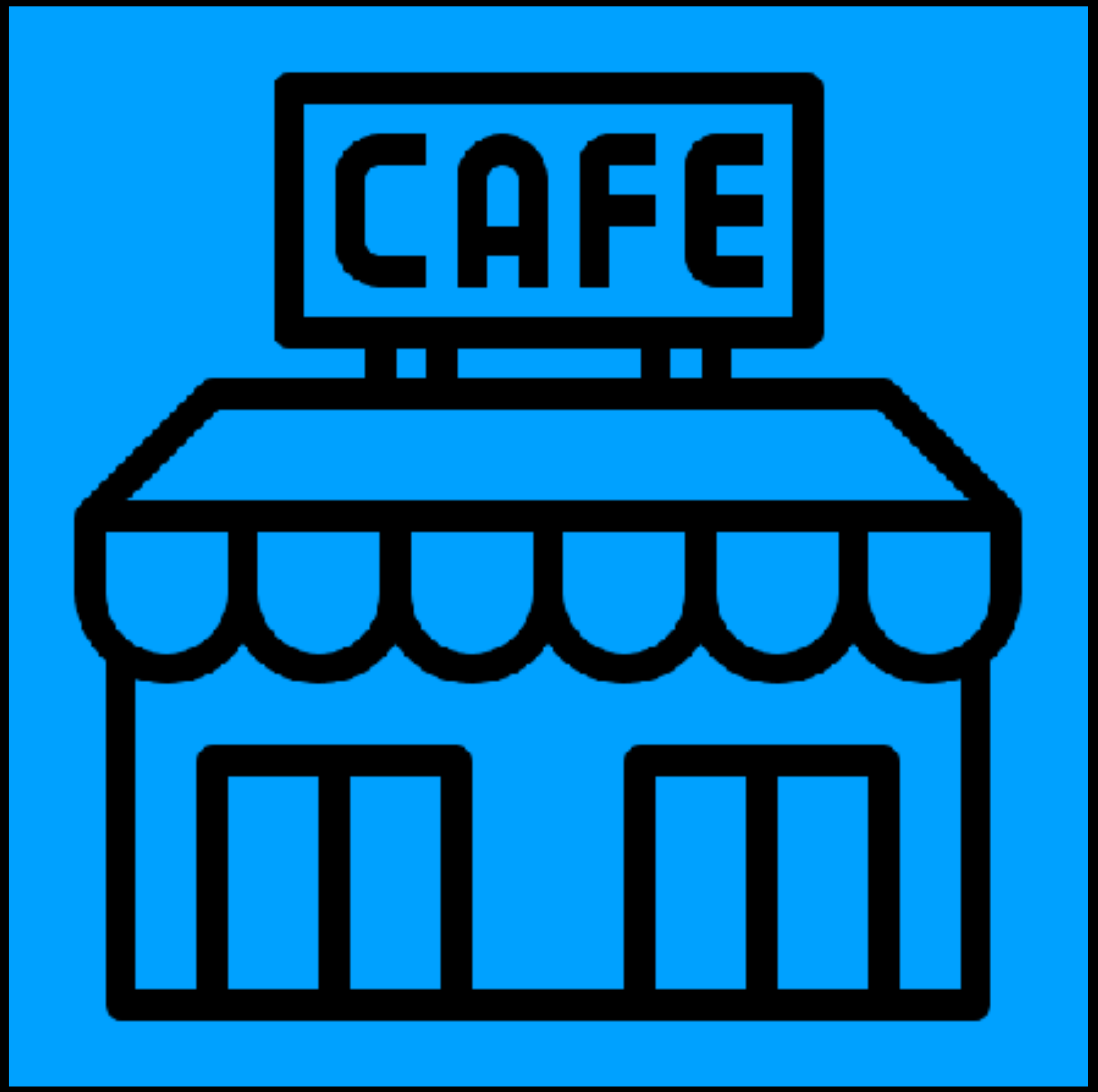
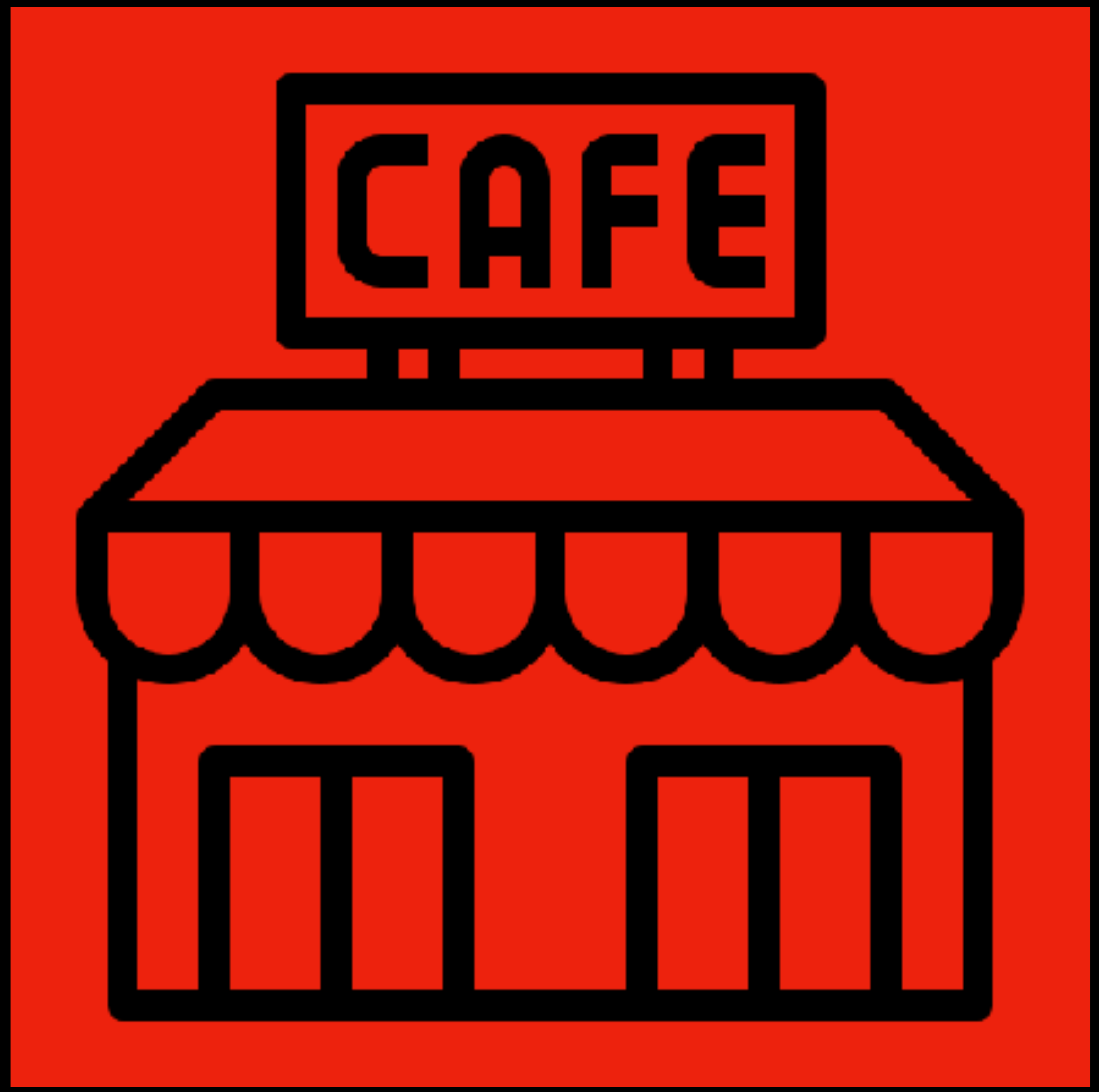
Options







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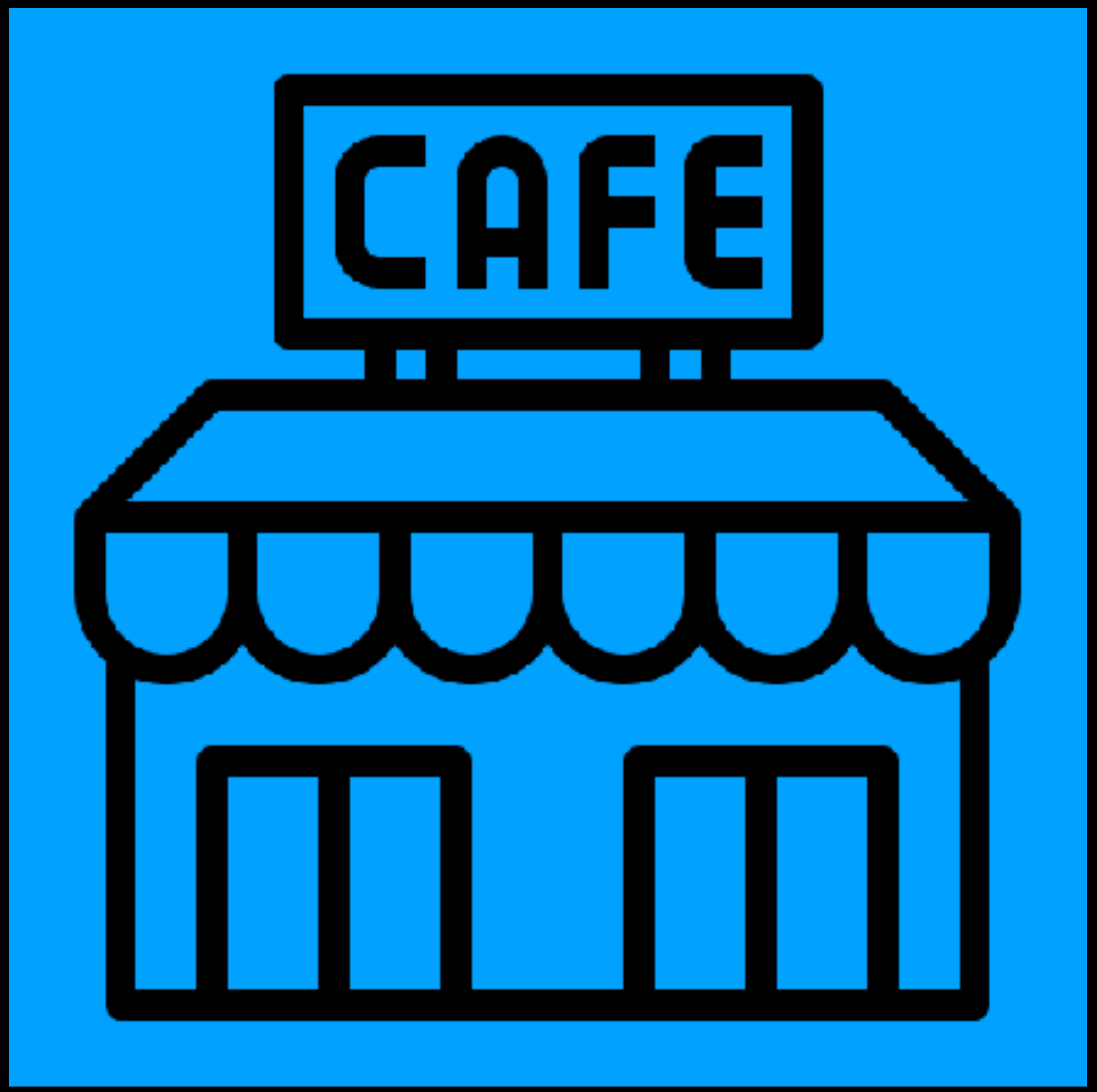
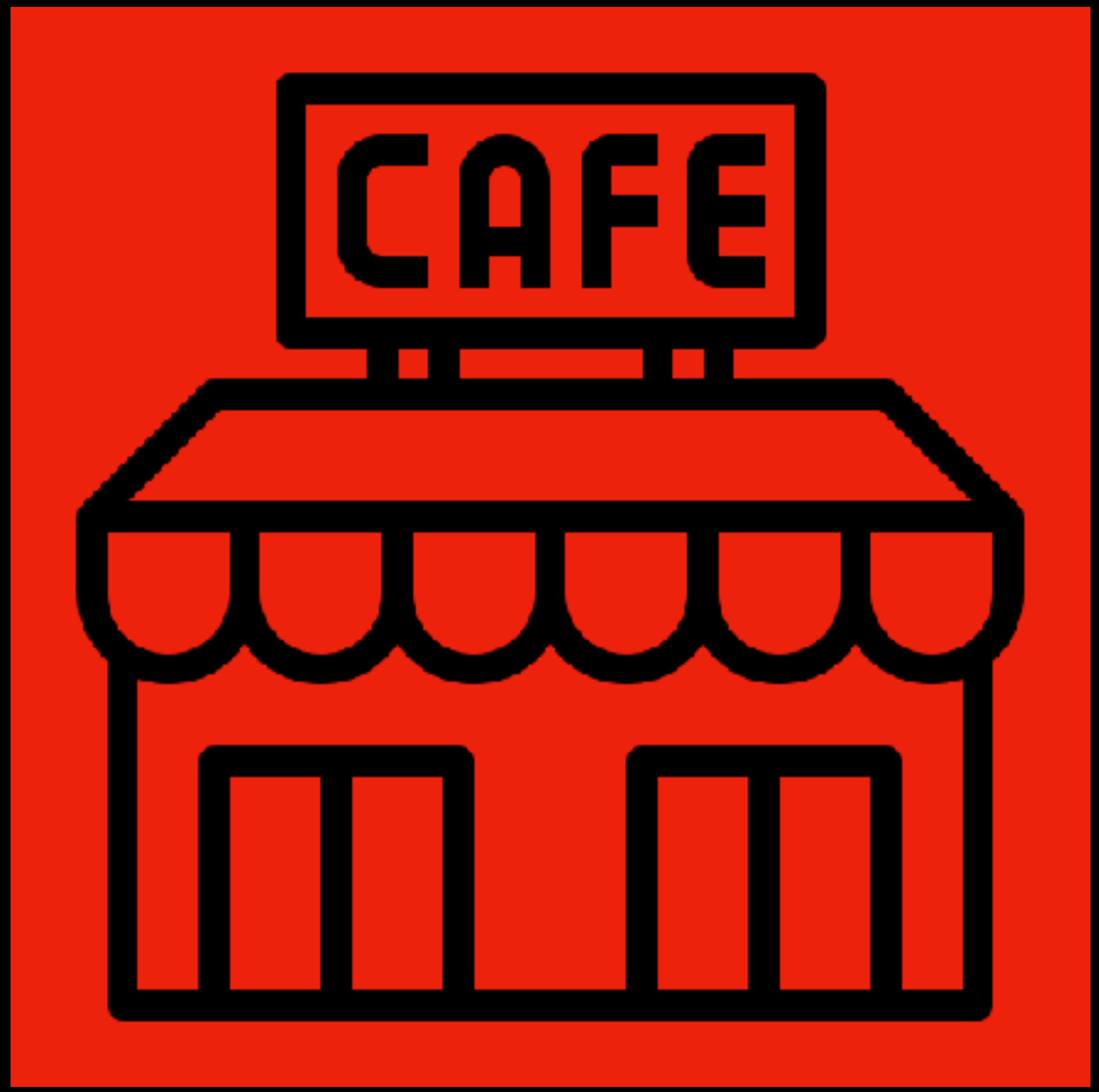


Outcomes



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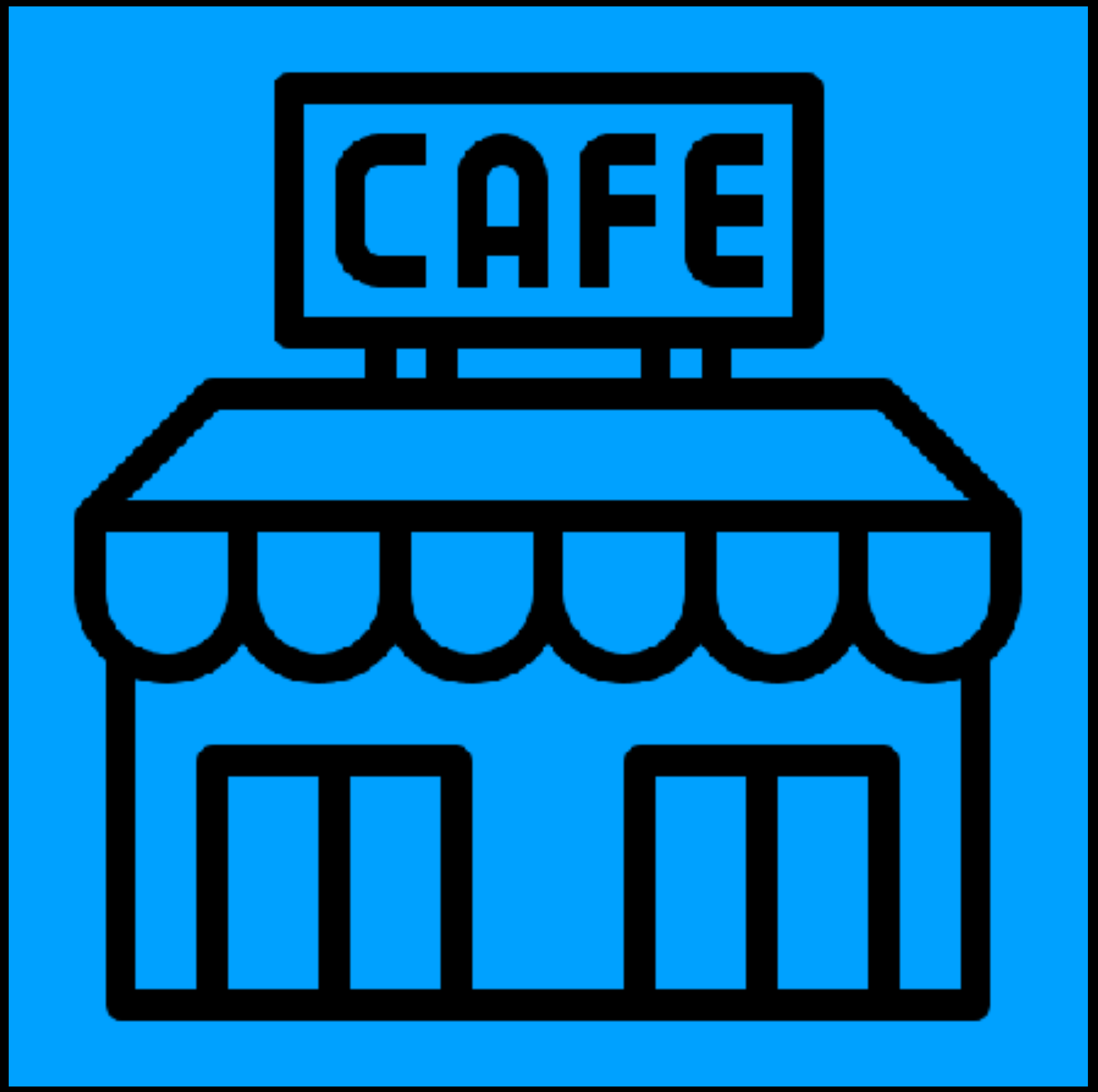

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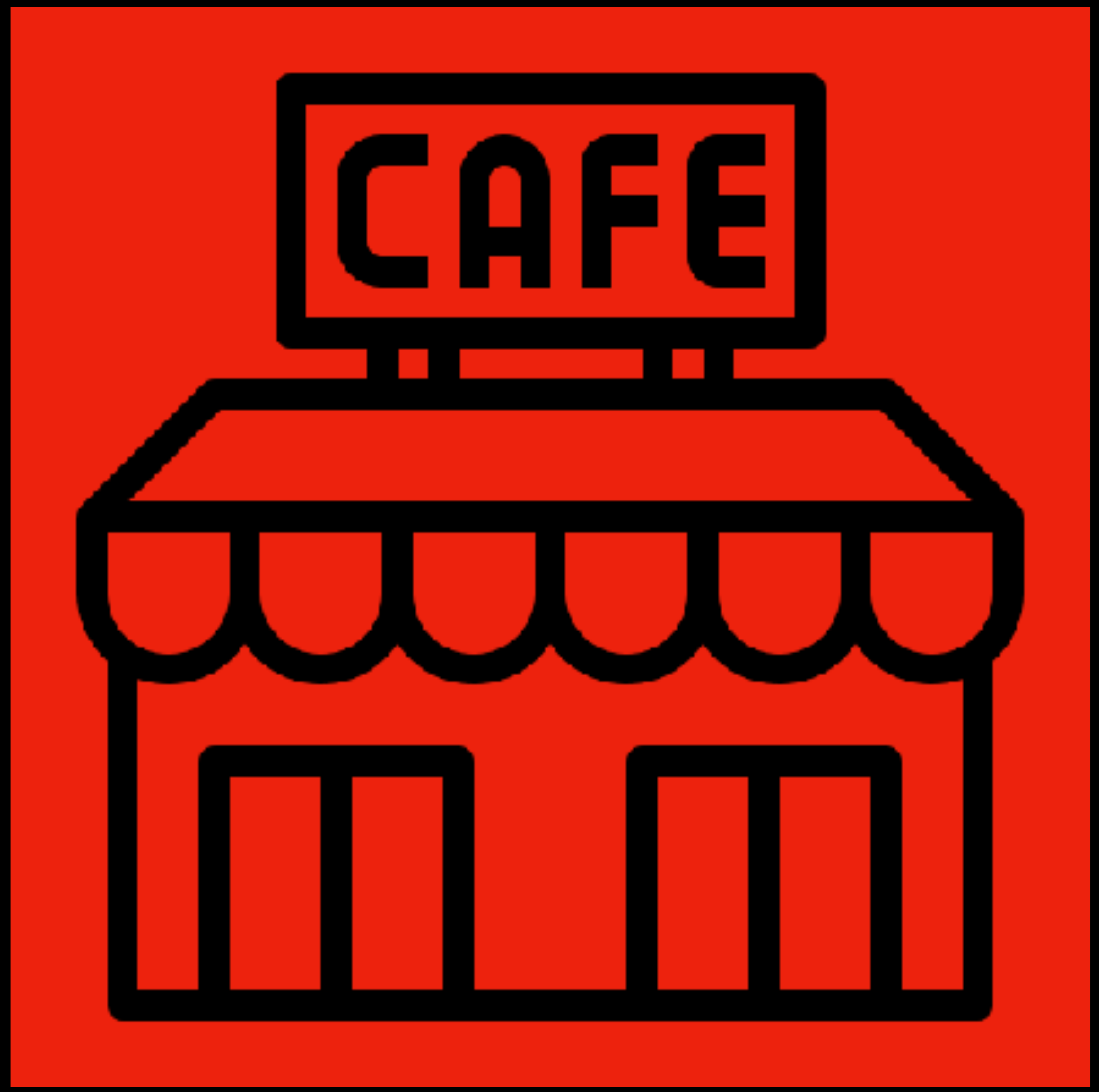

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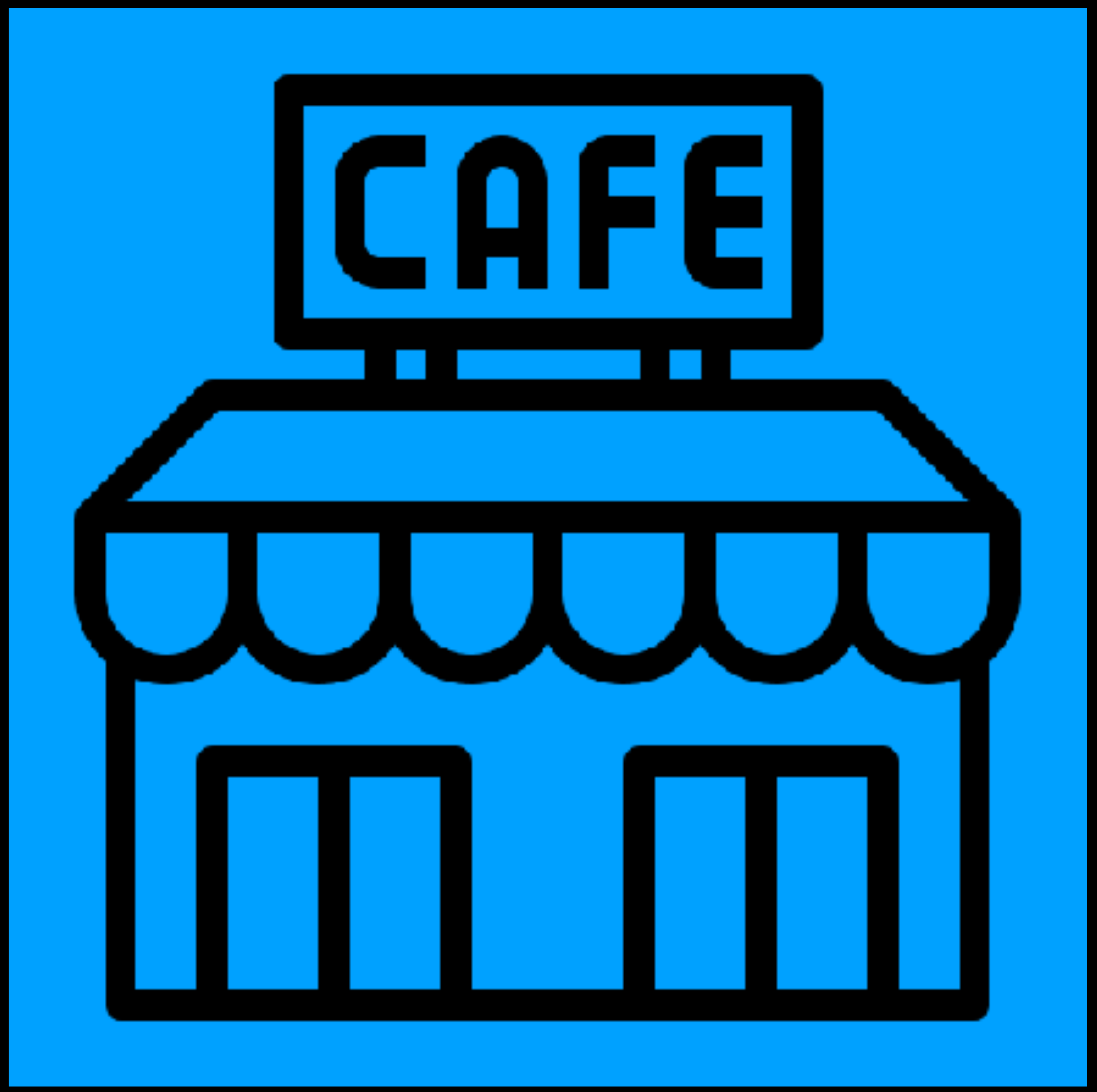

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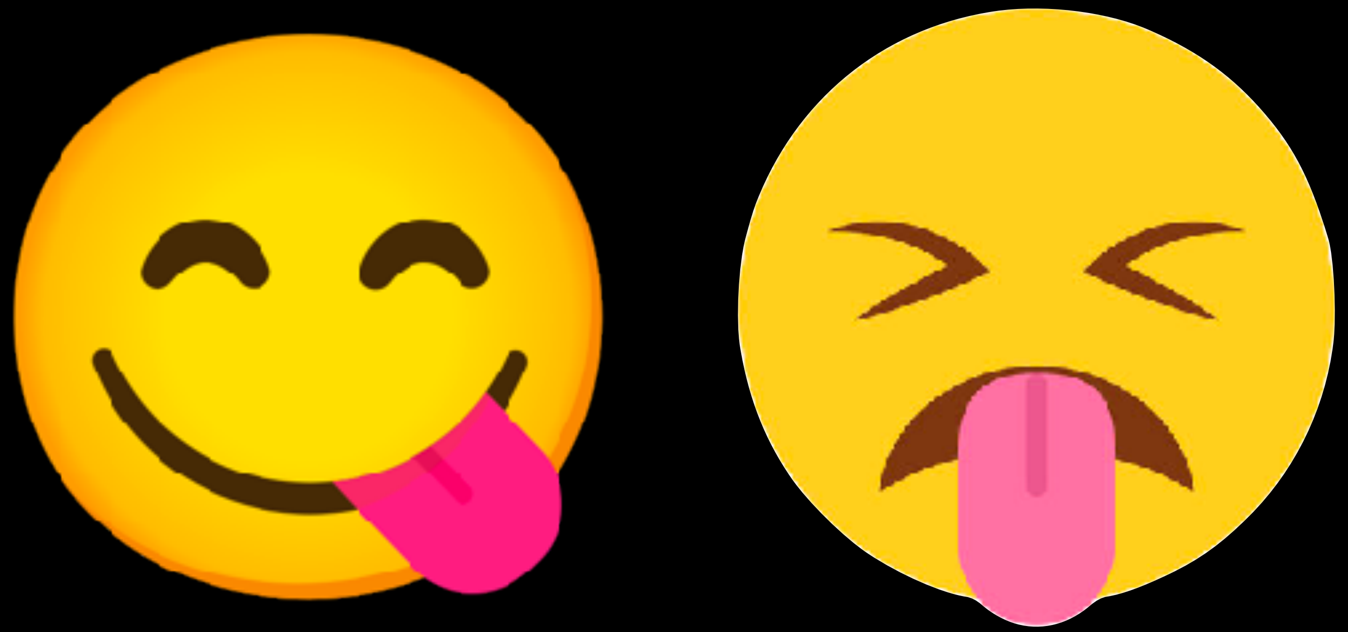
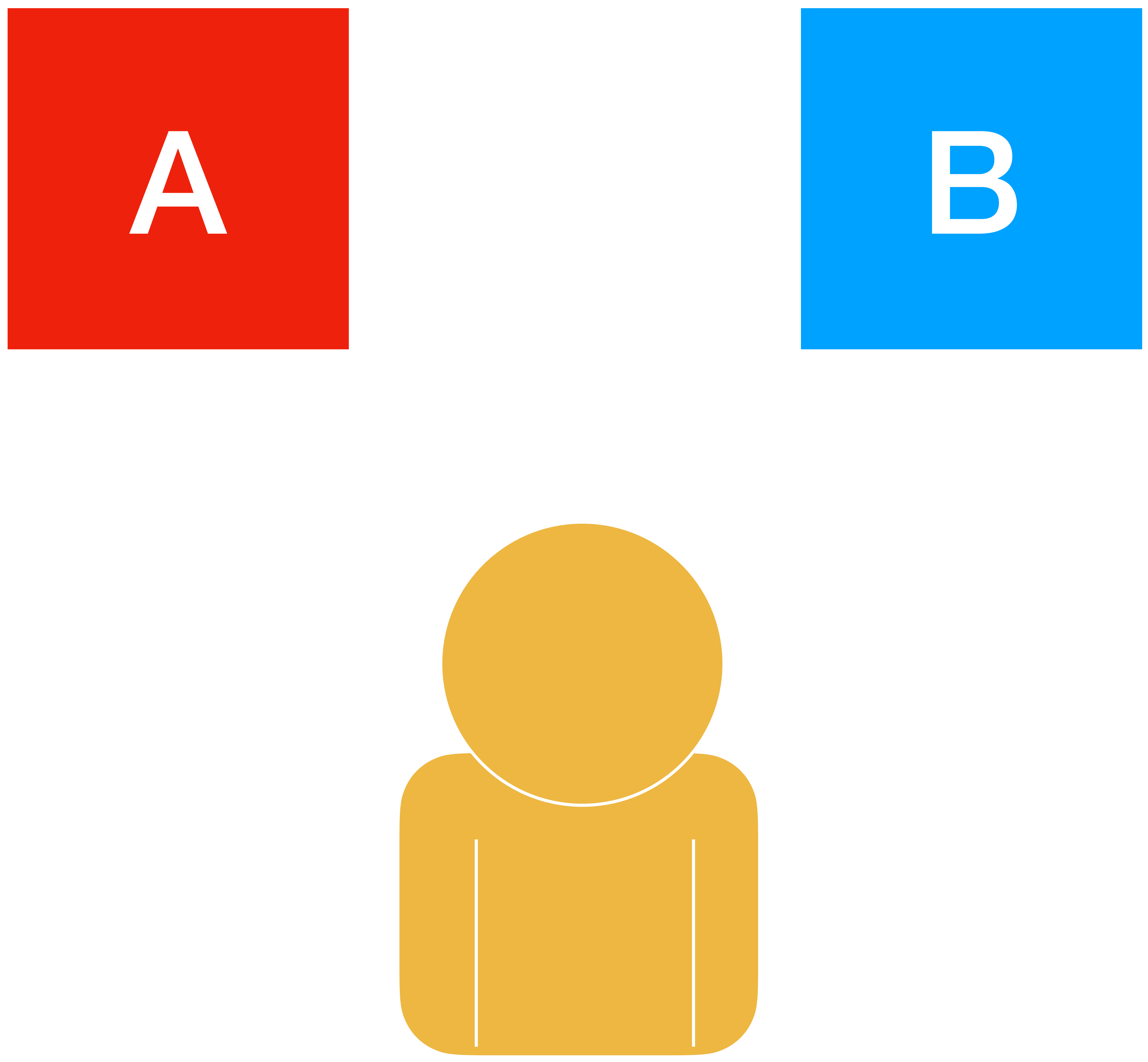


2-Armed Bandit Problem



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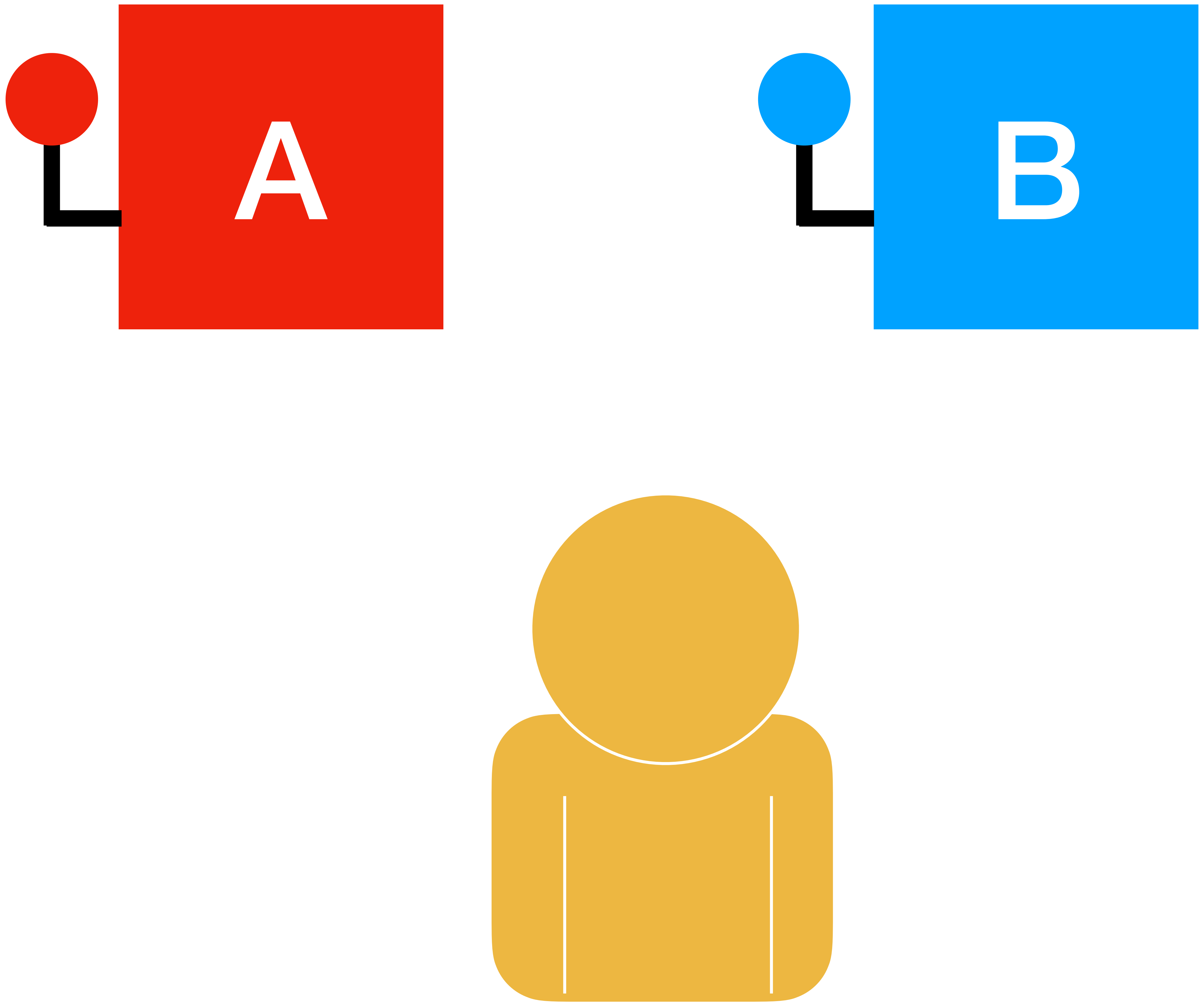

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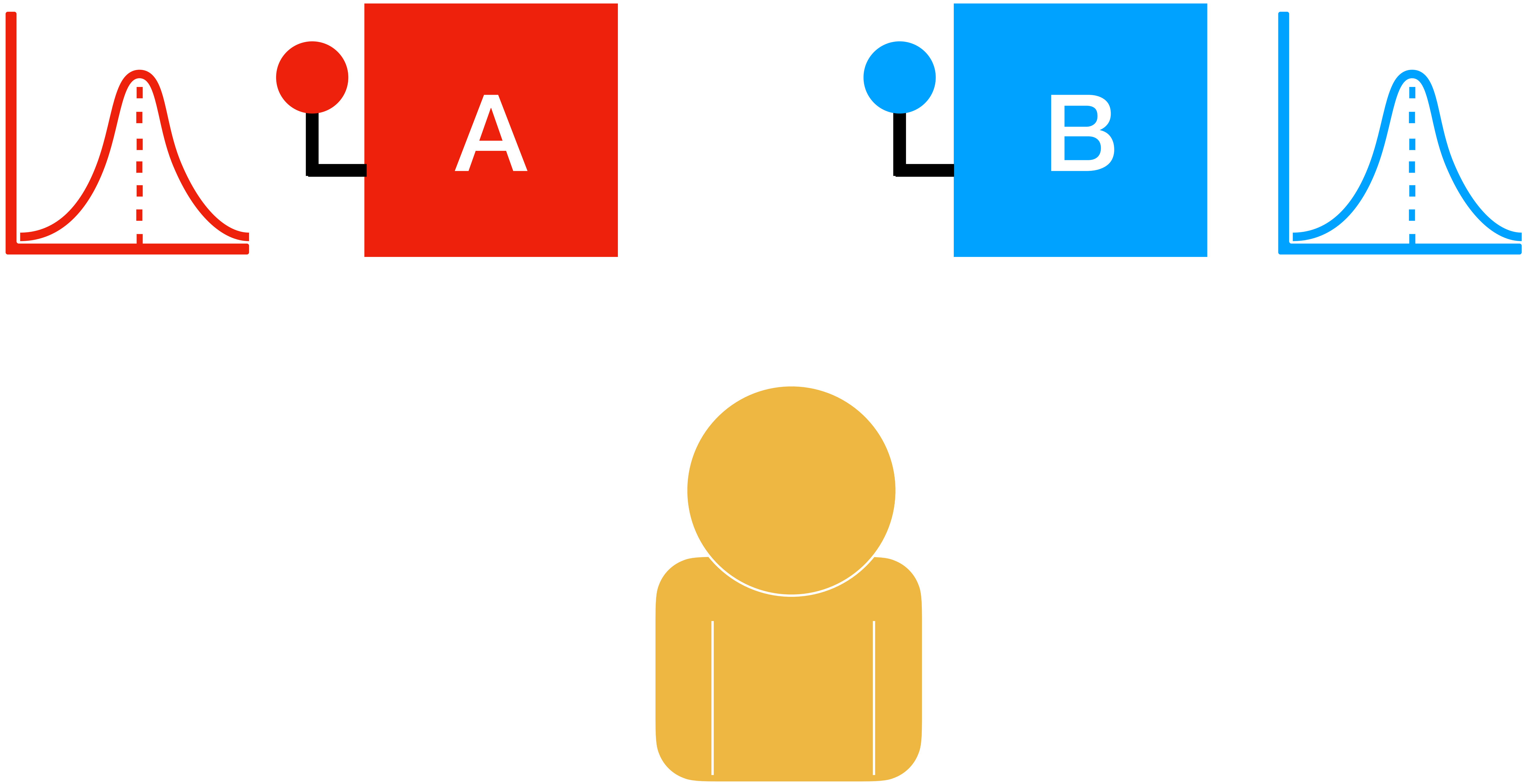

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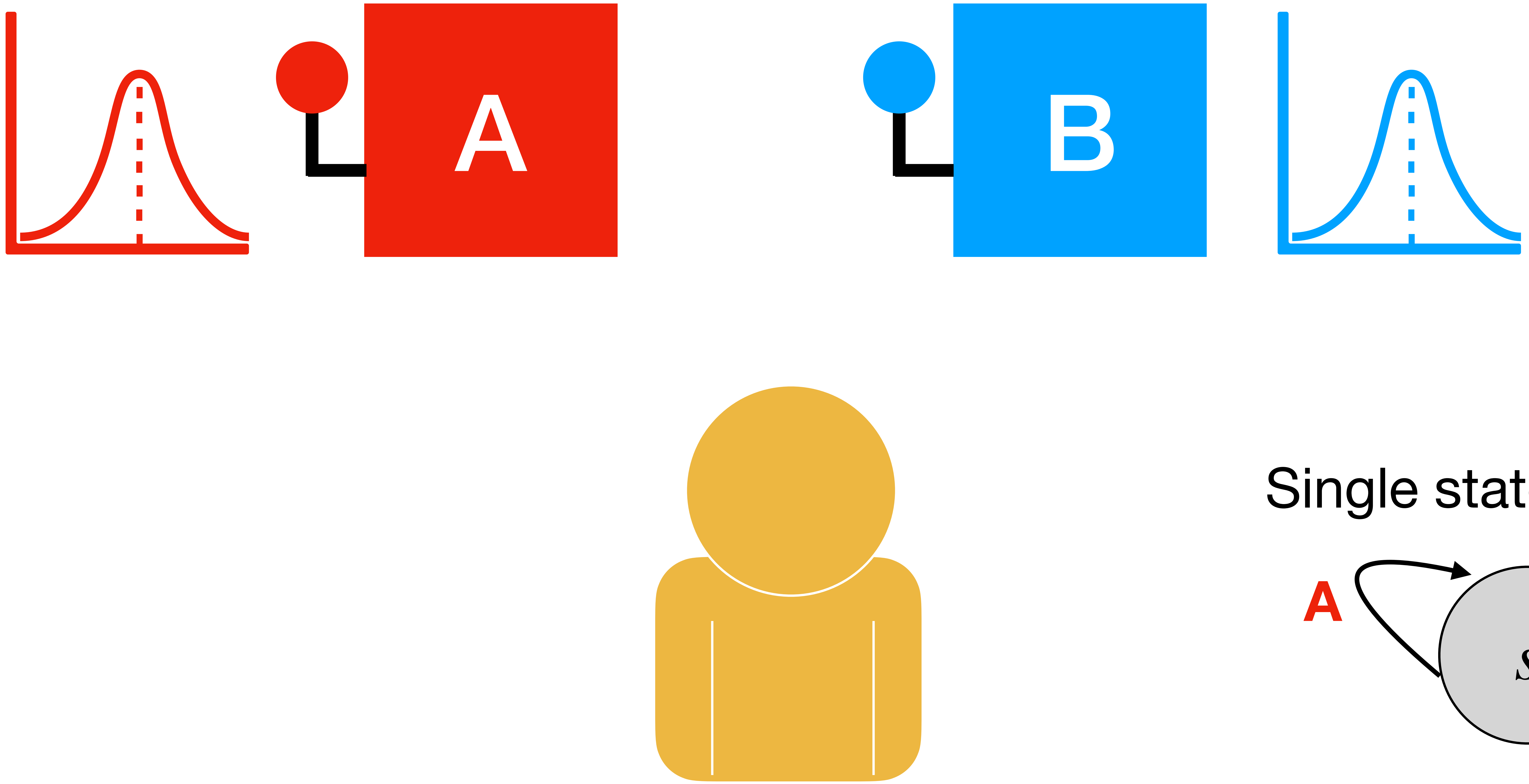
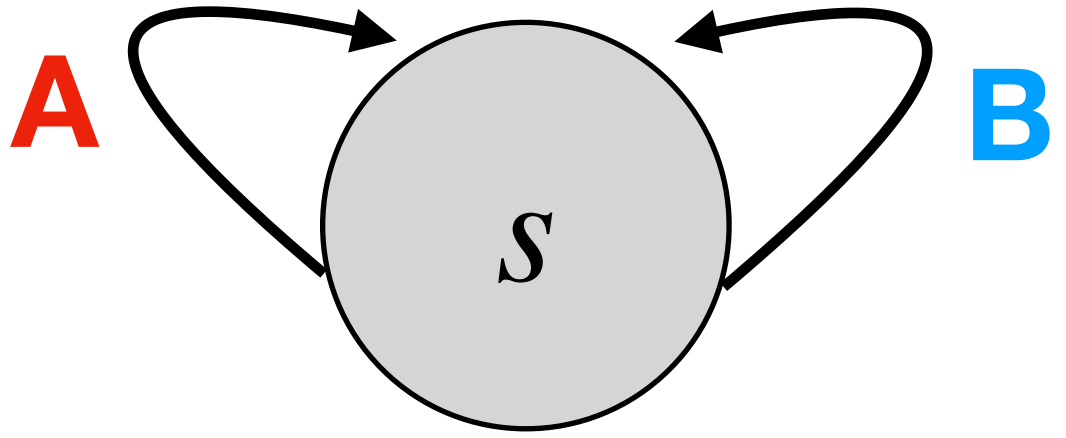


Single state problem



Q-Learning (Watkins, 1989)

Value learning

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$$Q_{t+1}(a) \leftarrow Q_t(a) + \eta \left[r - Q_t(a) \right]$$

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Observed reward

Predicted reward

δ

Reward prediction error (RPE)

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The delta-rule of learning:

- Learning occurs only when events violate expectations ($\delta \neq 0$)
- The magnitude of the error corresponds to how much we update our beliefs

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learning rate

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Exercise 1: Compute Q-values

A B

assume $\eta = .9$

	$Q(A)$	$Q(B)$	a	r	δ
t=1	0	0	A	5	
t=2			B	12	
t=3			B	4	
t=4			A	8	

Q-Learning (Watkins, 1989)

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$$Q_{t+1}(a) \leftarrow Q_t(a) + \underbrace{\eta}_{\text{learning rate}} \underbrace{[\underbrace{r}_{\text{Observed reward}} - \underbrace{Q_t(a)}_{\text{Predicted reward}}]}_{\delta \text{ Reward prediction error (RPE)}}$$

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	$Q(A)$	$Q(B)$	a	r	δ
t=1	0	0	A	5	5
t=2	4.5	0	B	12	12
t=3	4.5	10.8	B	4	-6.8
t=4	4.5	4.68	A	8	3.5

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$$P(a) \propto \exp(Q_t(a)/\tau)$$

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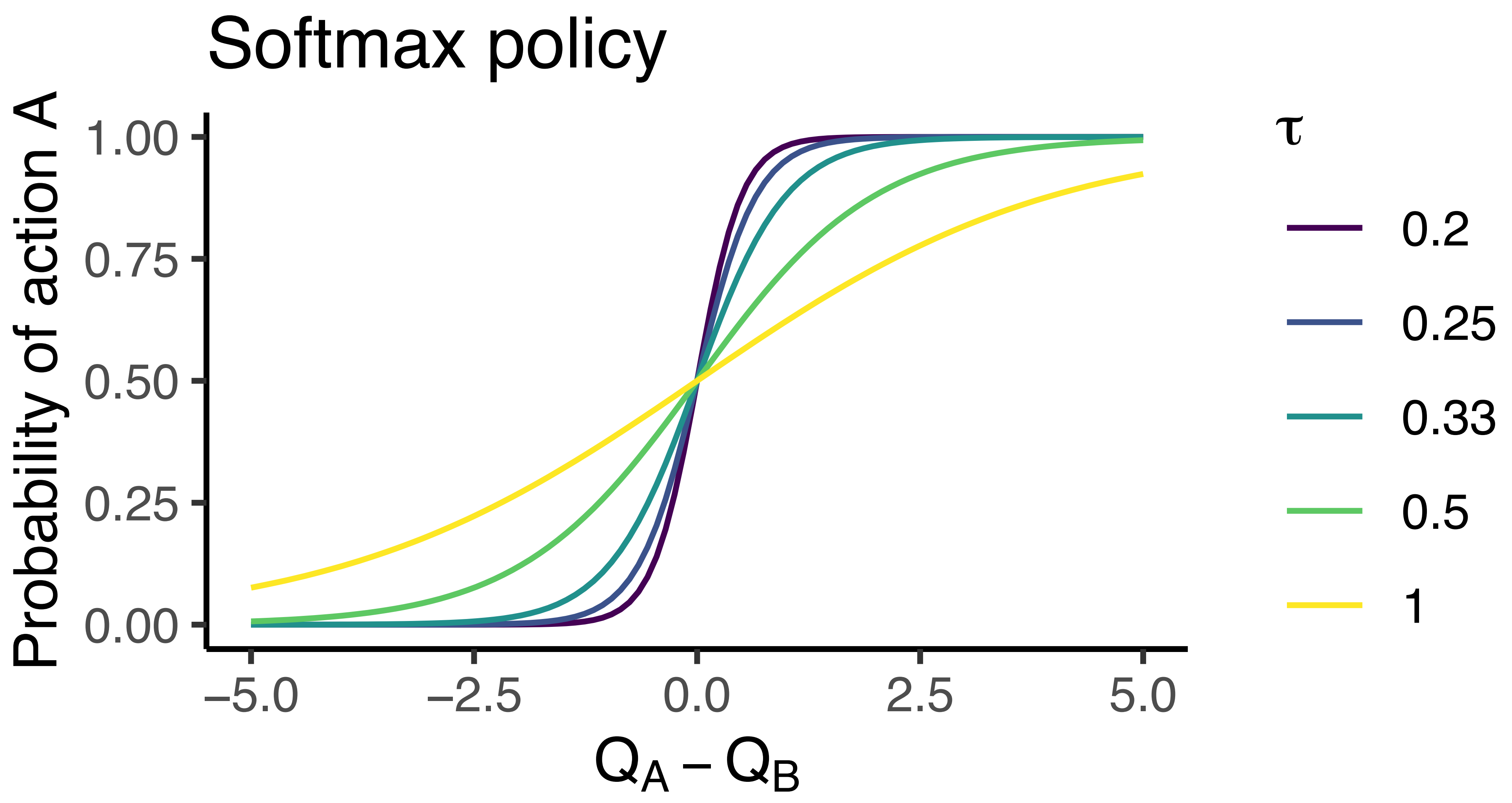

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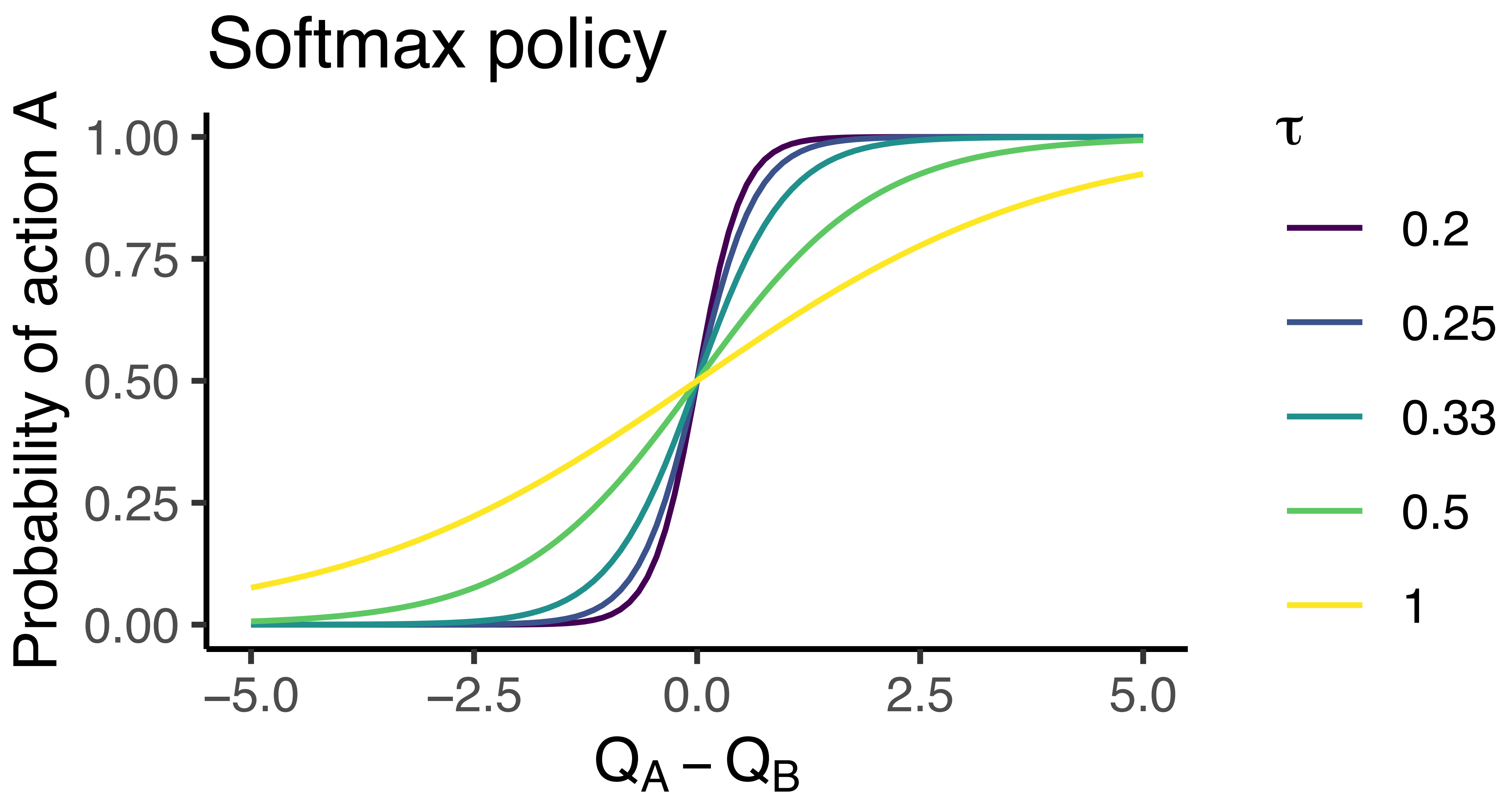

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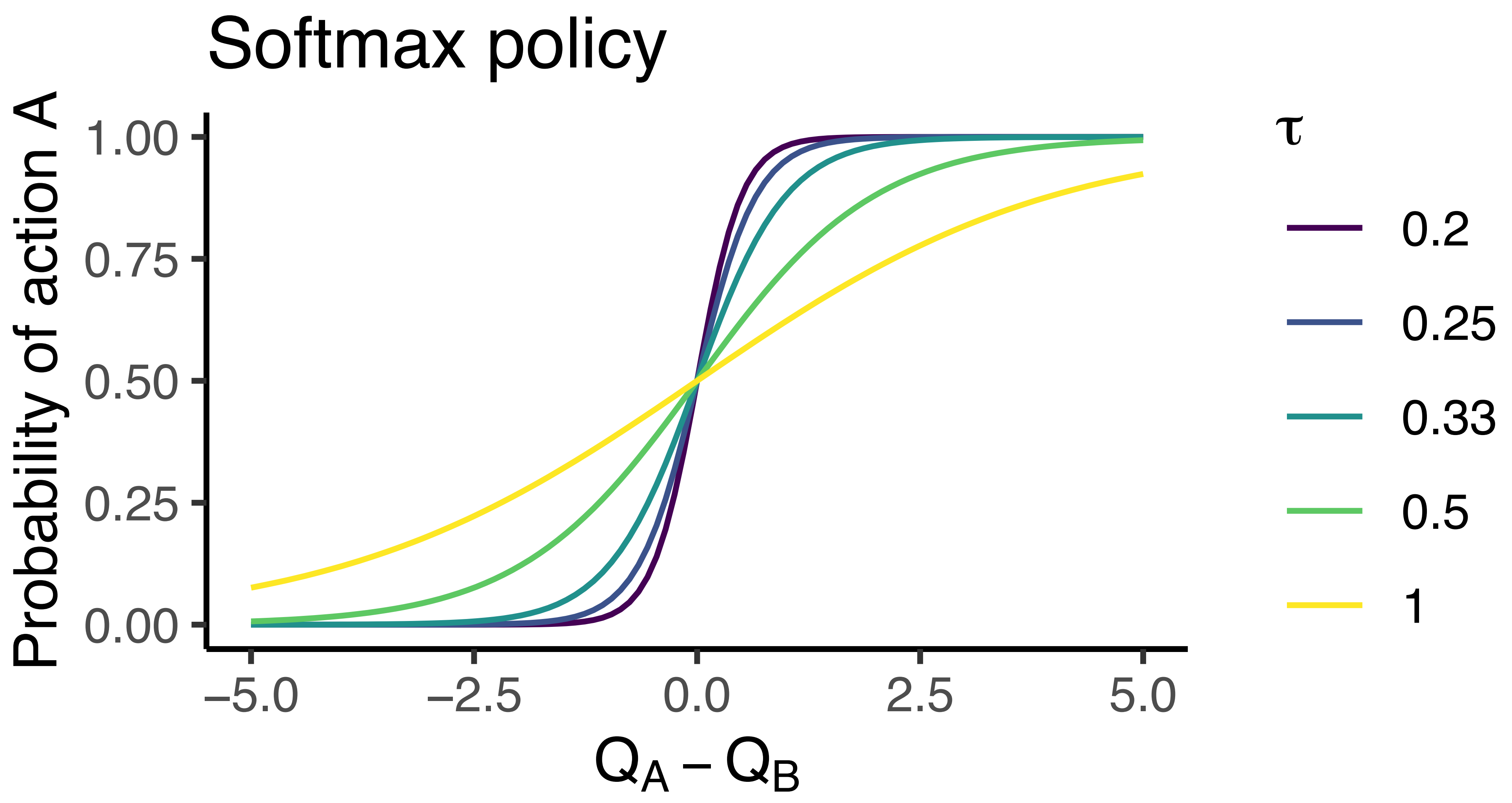


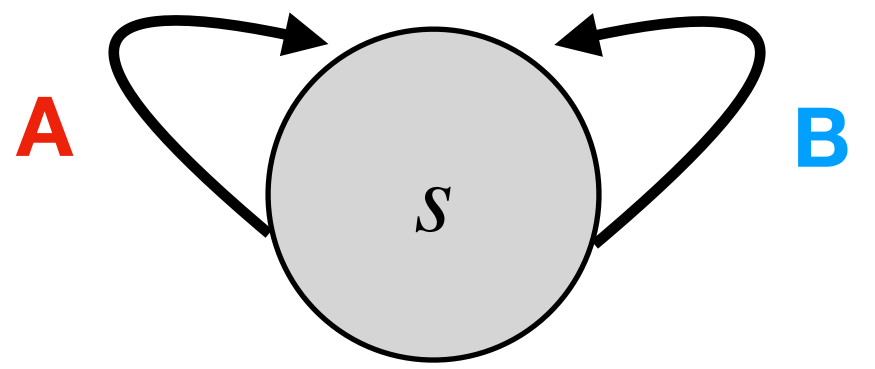
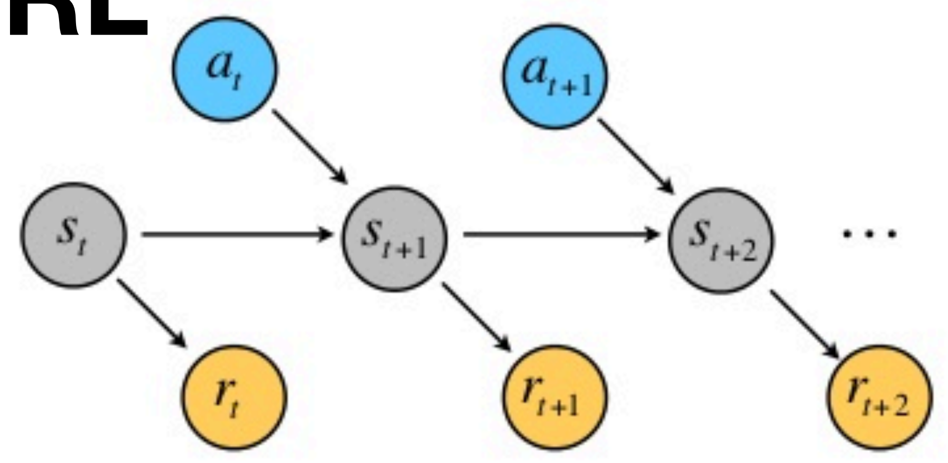
Exercise 2: Sample actions from policy

A B

assume:
$\eta = .9$
$\tau = .25$

	$Q(A)$	$Q(B)$	a	r	δ
t=1	0	0			
t=2					
t=3					
t=4					

Moving back to the general RL problem

	Bandit problem 	**General RL** 
Examples	choosing restaurants, buying a phone, funding research, A/B testing of advertisements, ….	robot bartender, playing games, self-driving car, chatbots, etc…
Value representations	$Q(a)$	$Q(a, s)$ or $V(s)$
Policy	$\pi(a)$	$\pi(a \mid s)$
Planning?	Not needed	Important!
Long-term Value?	$V(a) = Q(a)$	

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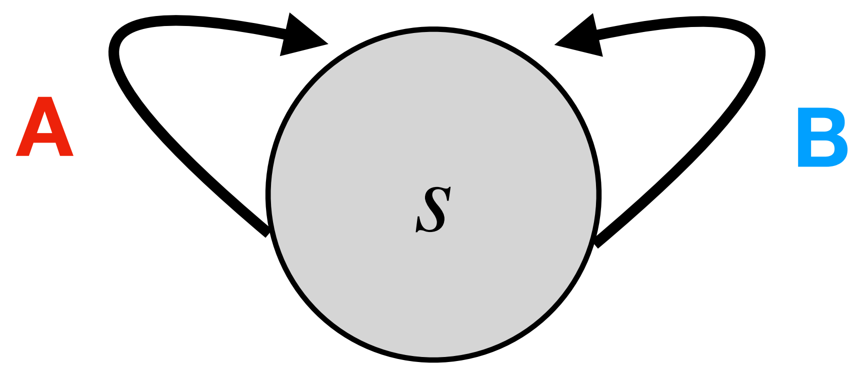
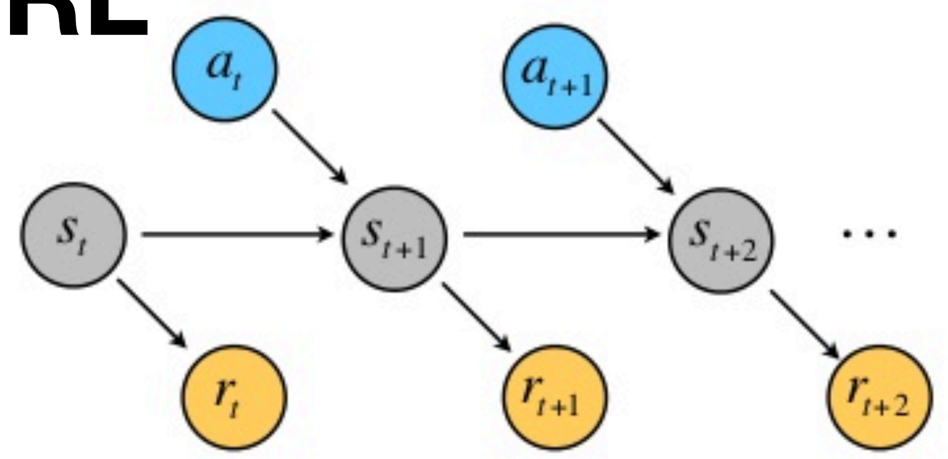
	Bandit problem 	**General RL** 
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- How do we assign credit to actions that are responsible for future reward? (Minsky, 1961)

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$$V(s) \leftarrow V(s) + \eta \left(r + \underbrace{\gamma V(s')}_{\text{discounted future value}} - V(s) \right)$$

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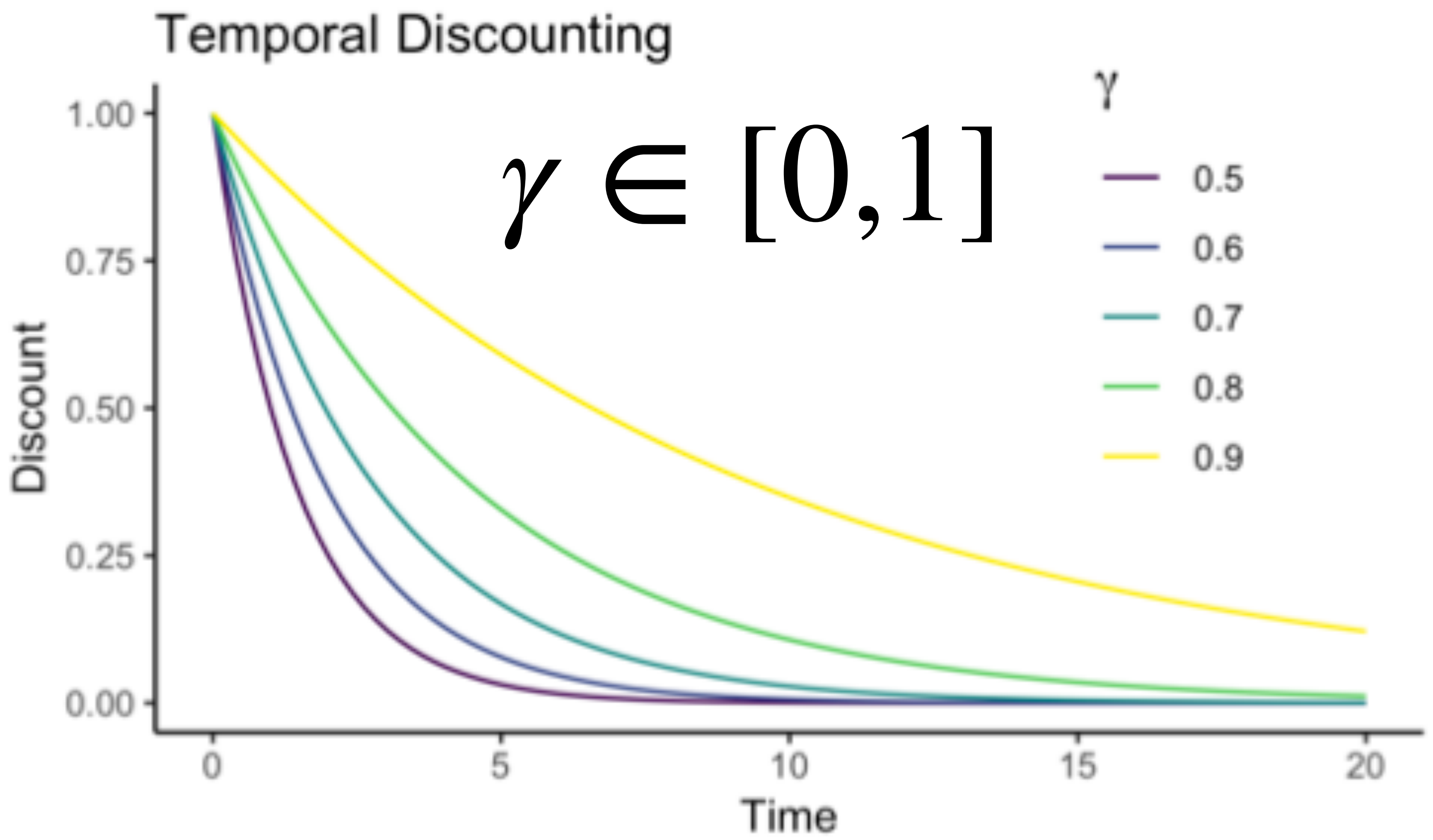

$$\gamma \in [0,1]$$

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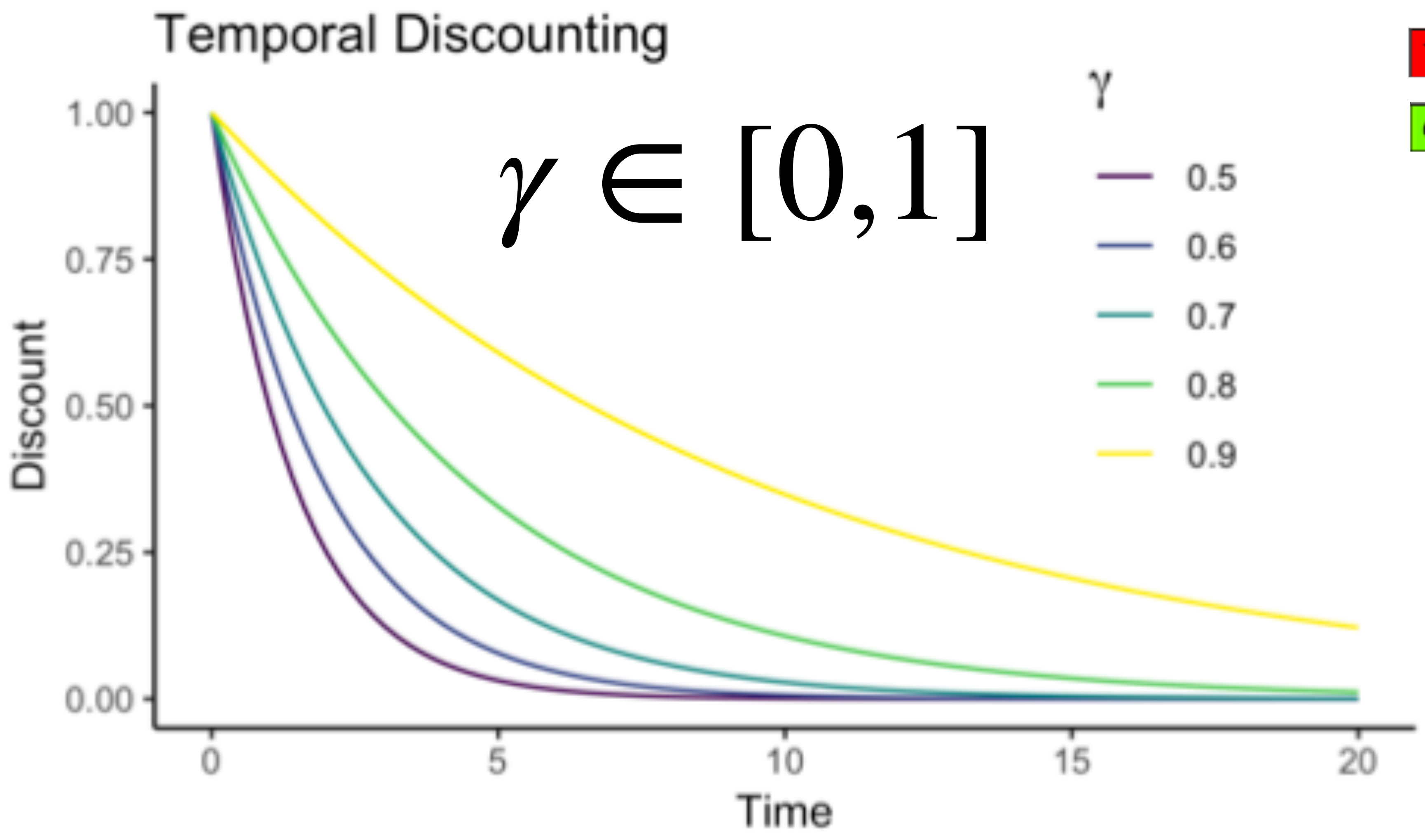

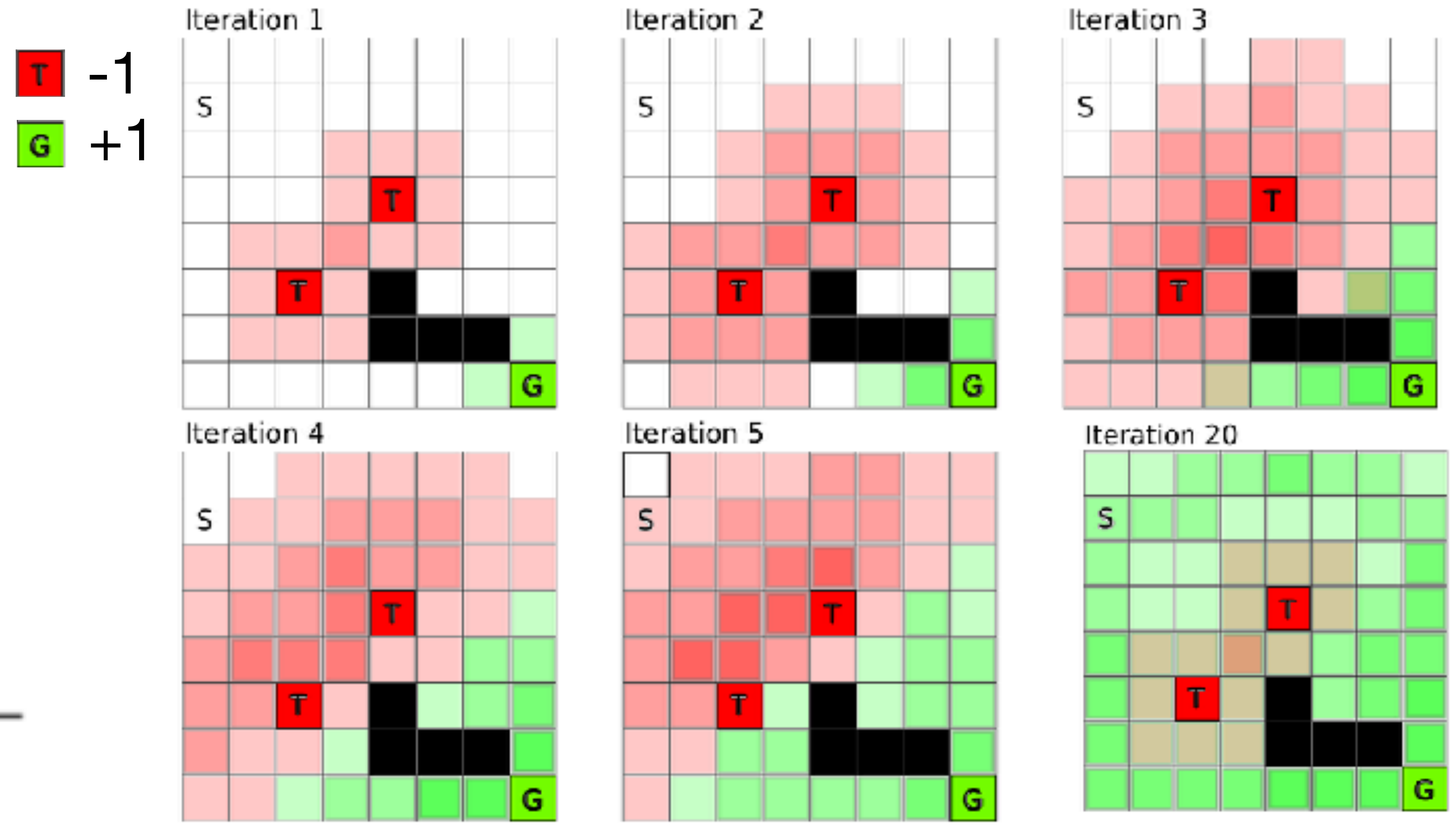

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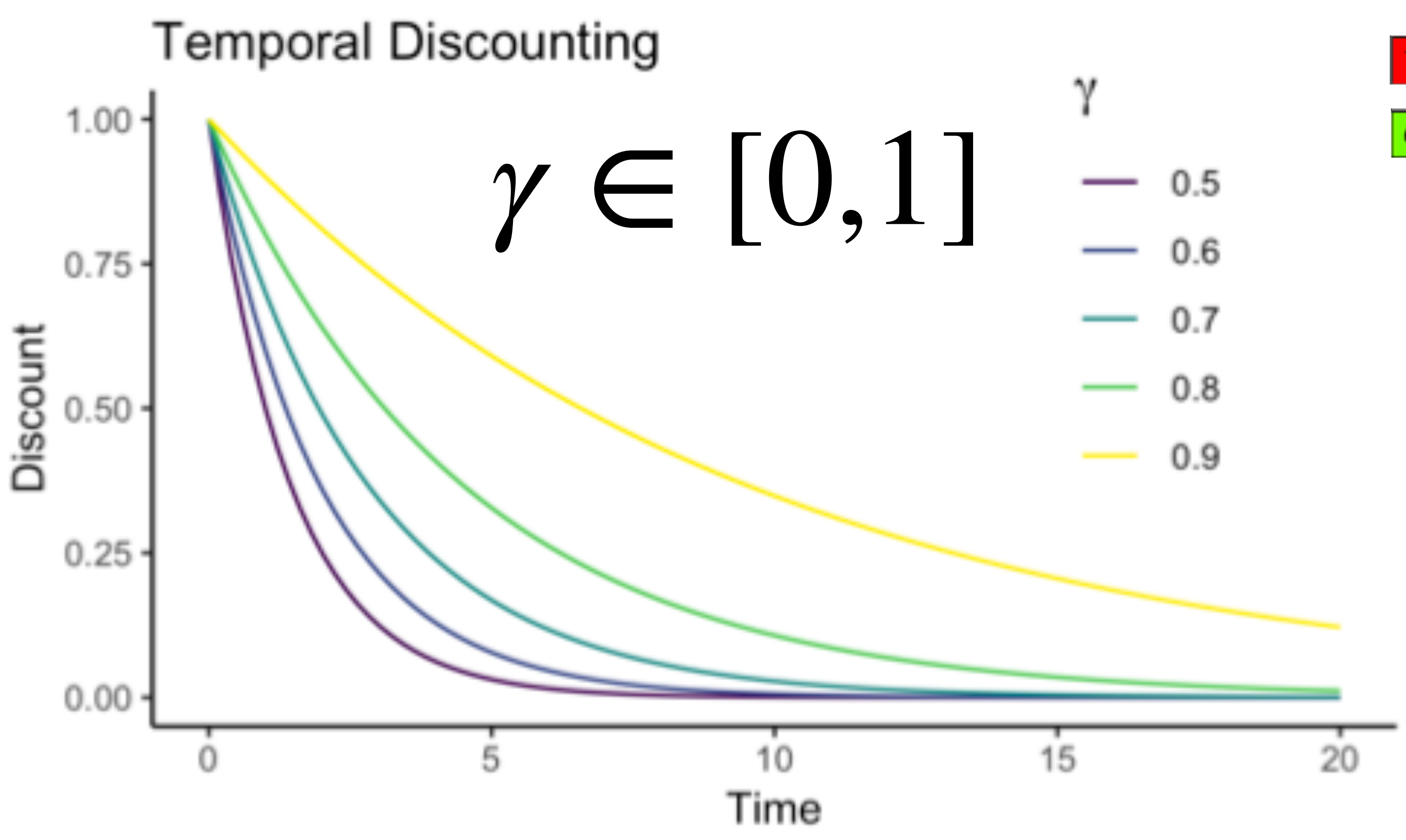
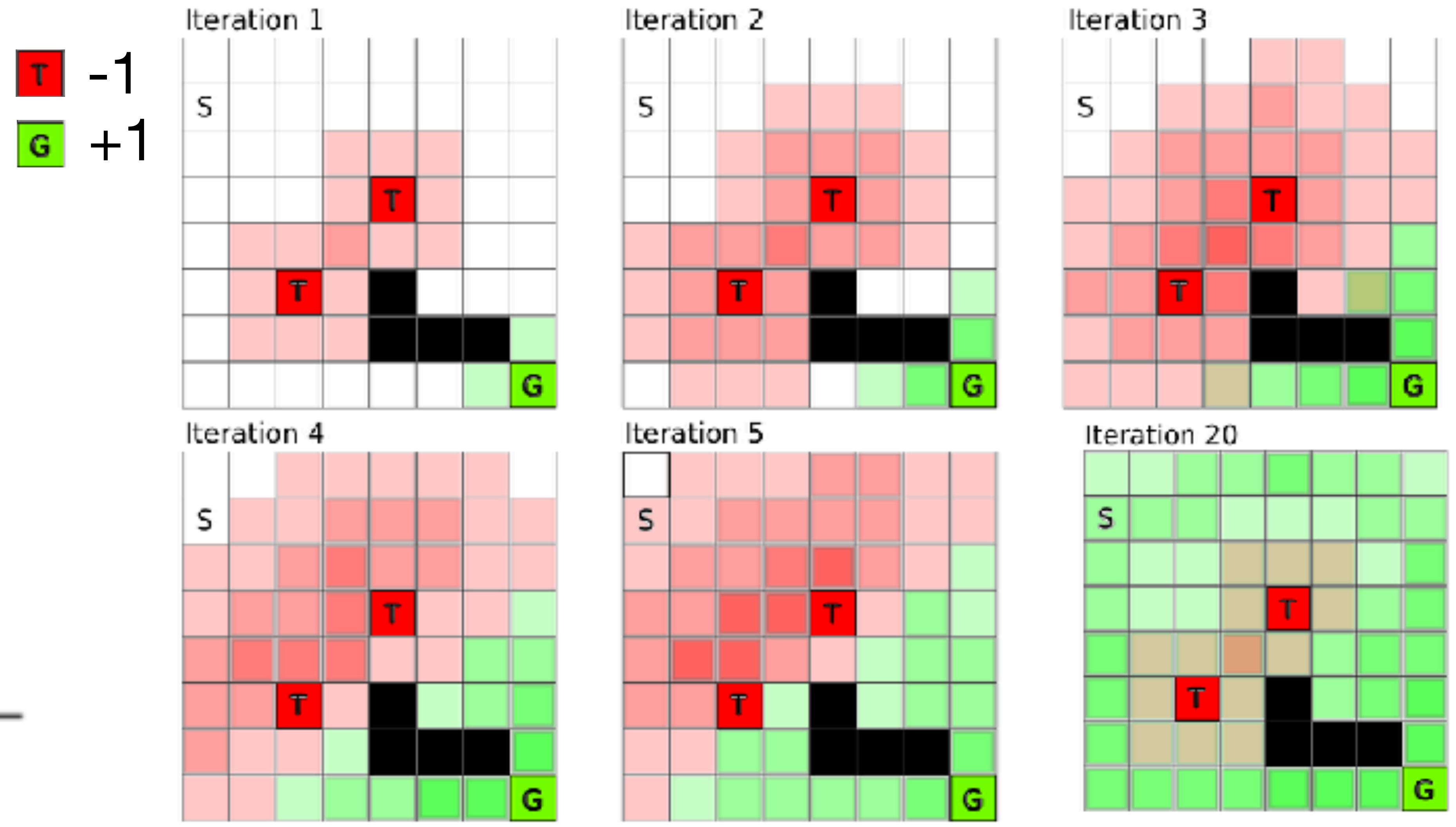
TD Prediction Error

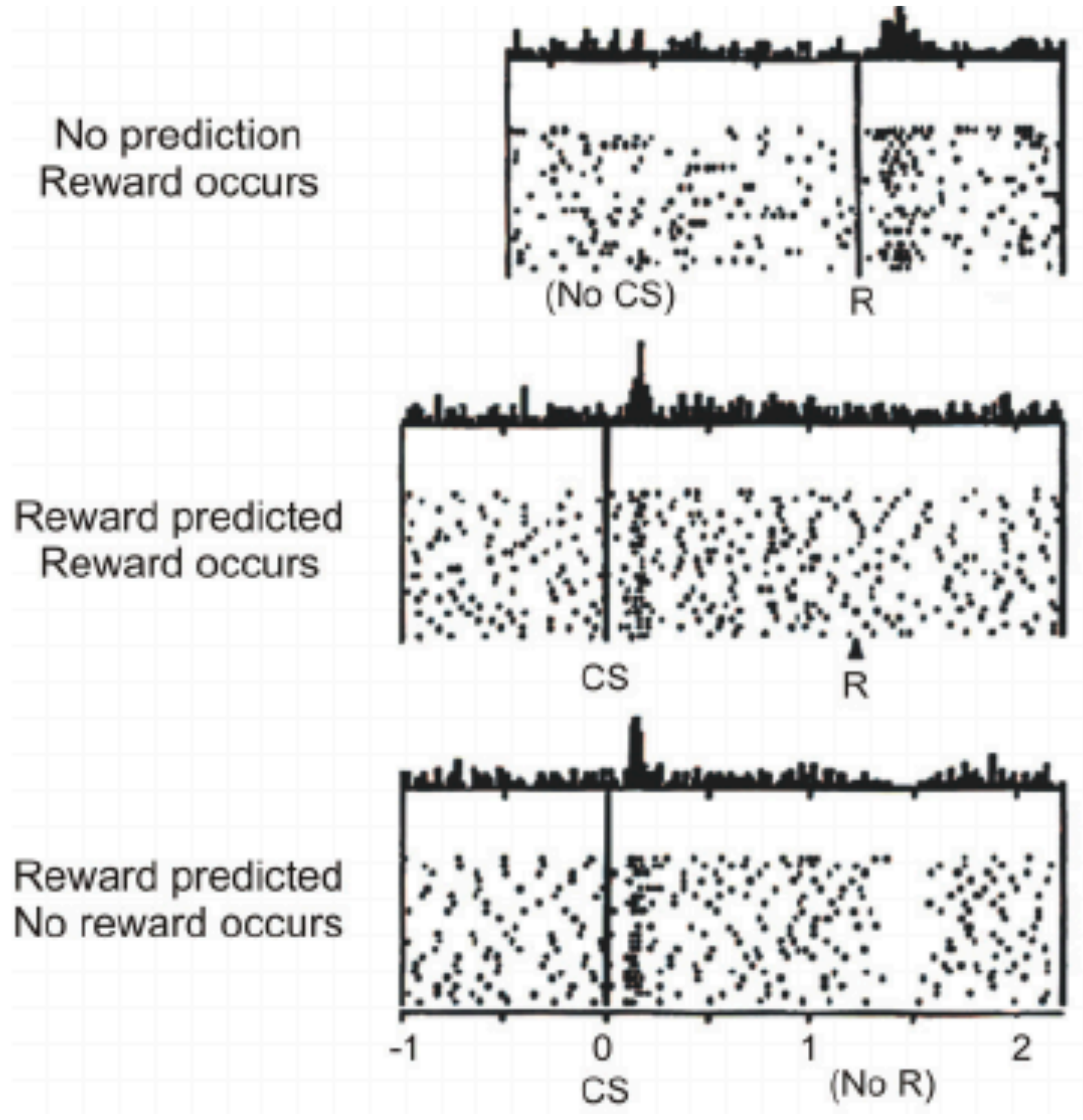
Schultz et al. (*Science* 1997)

Dopaminergic Neurons

$$\gamma \in [0,1]$$







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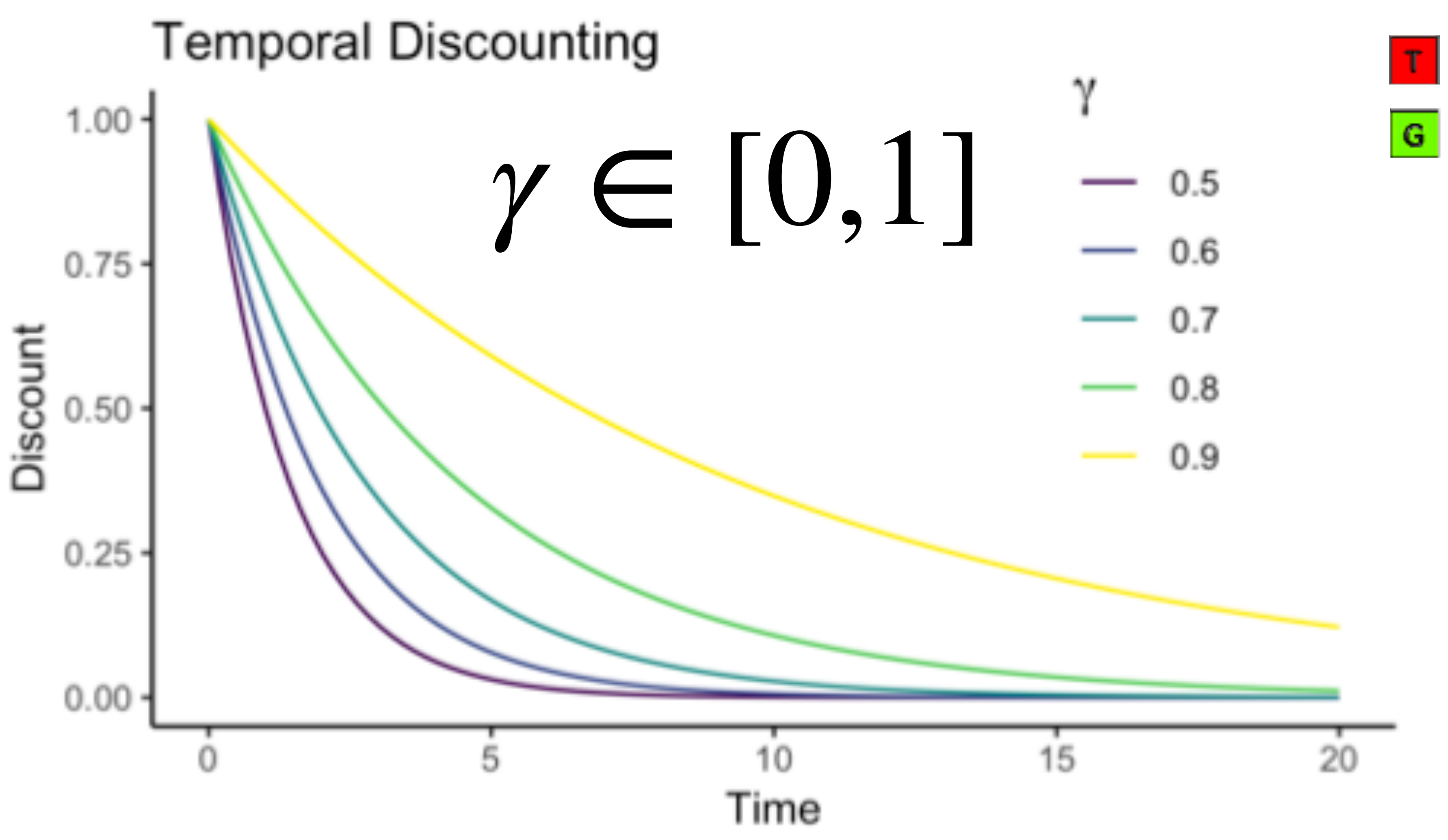

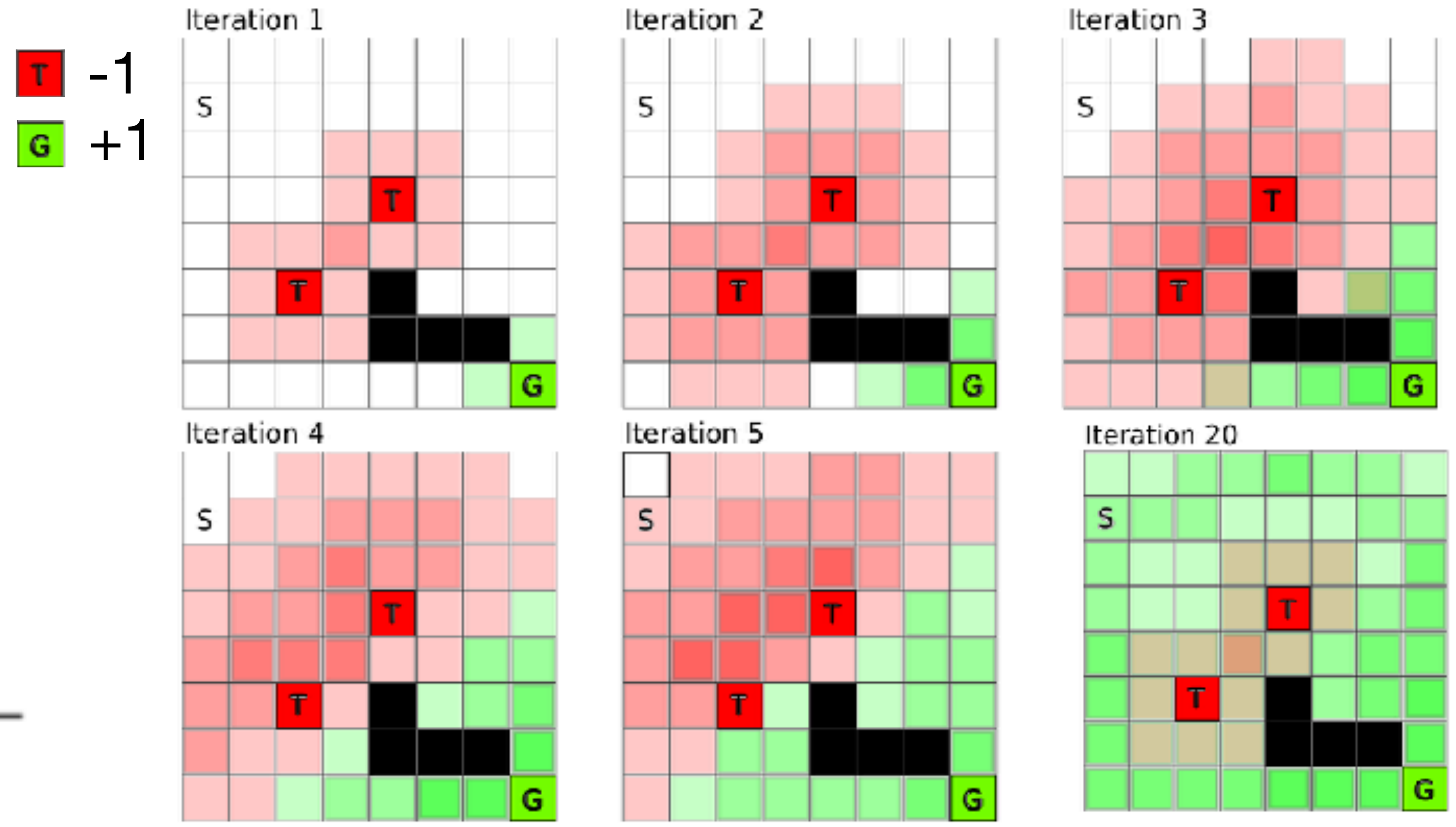

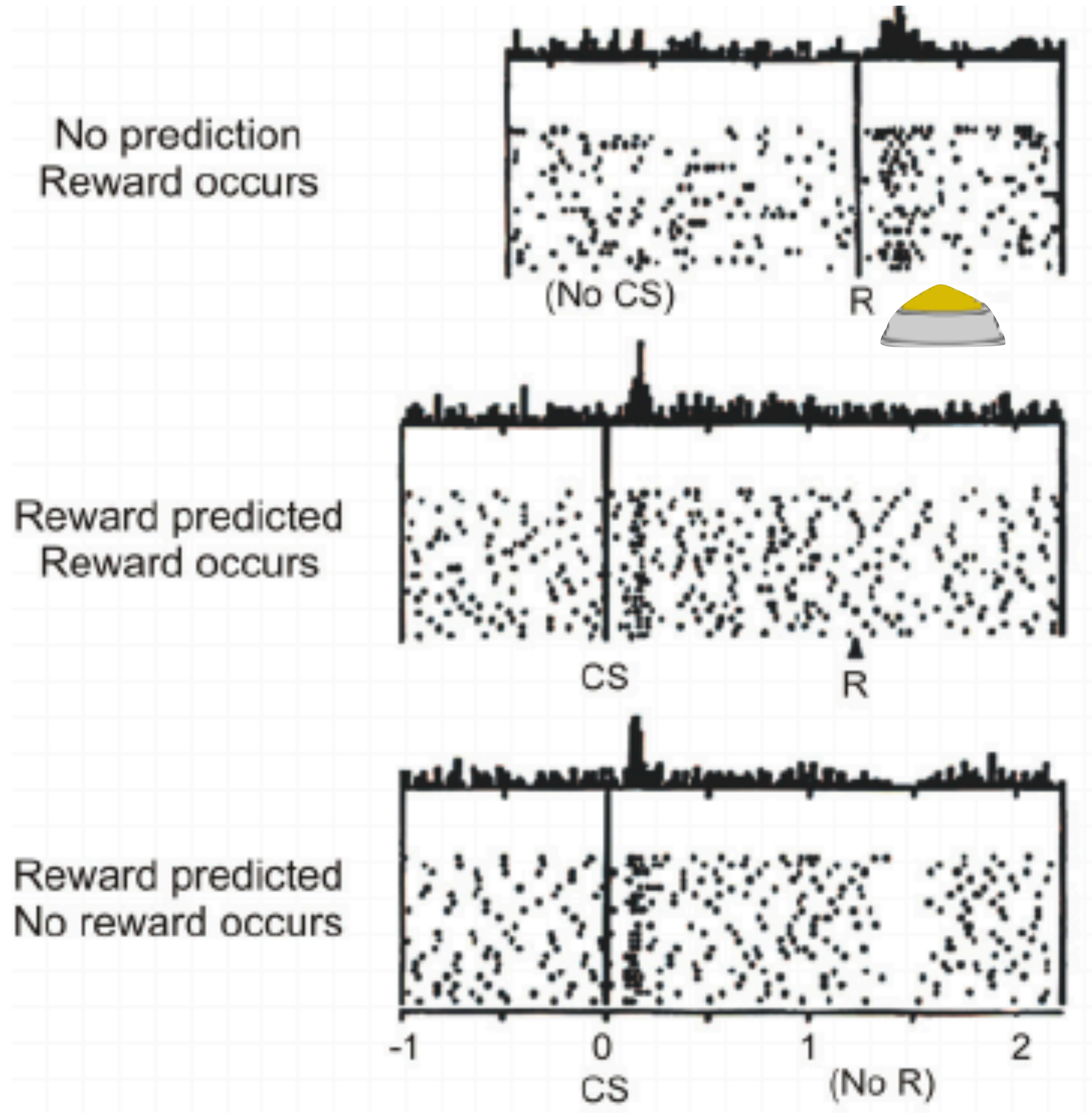

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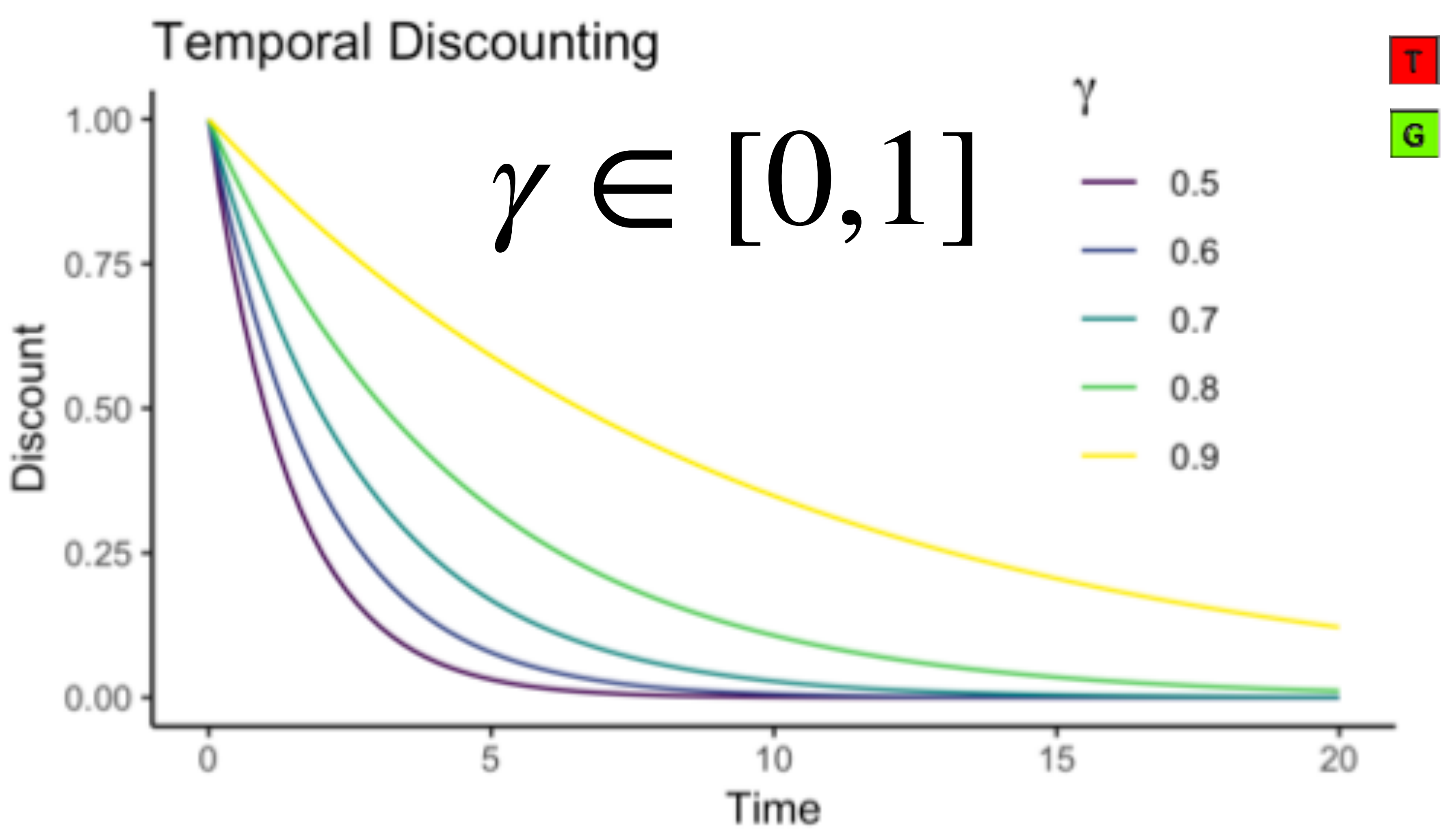

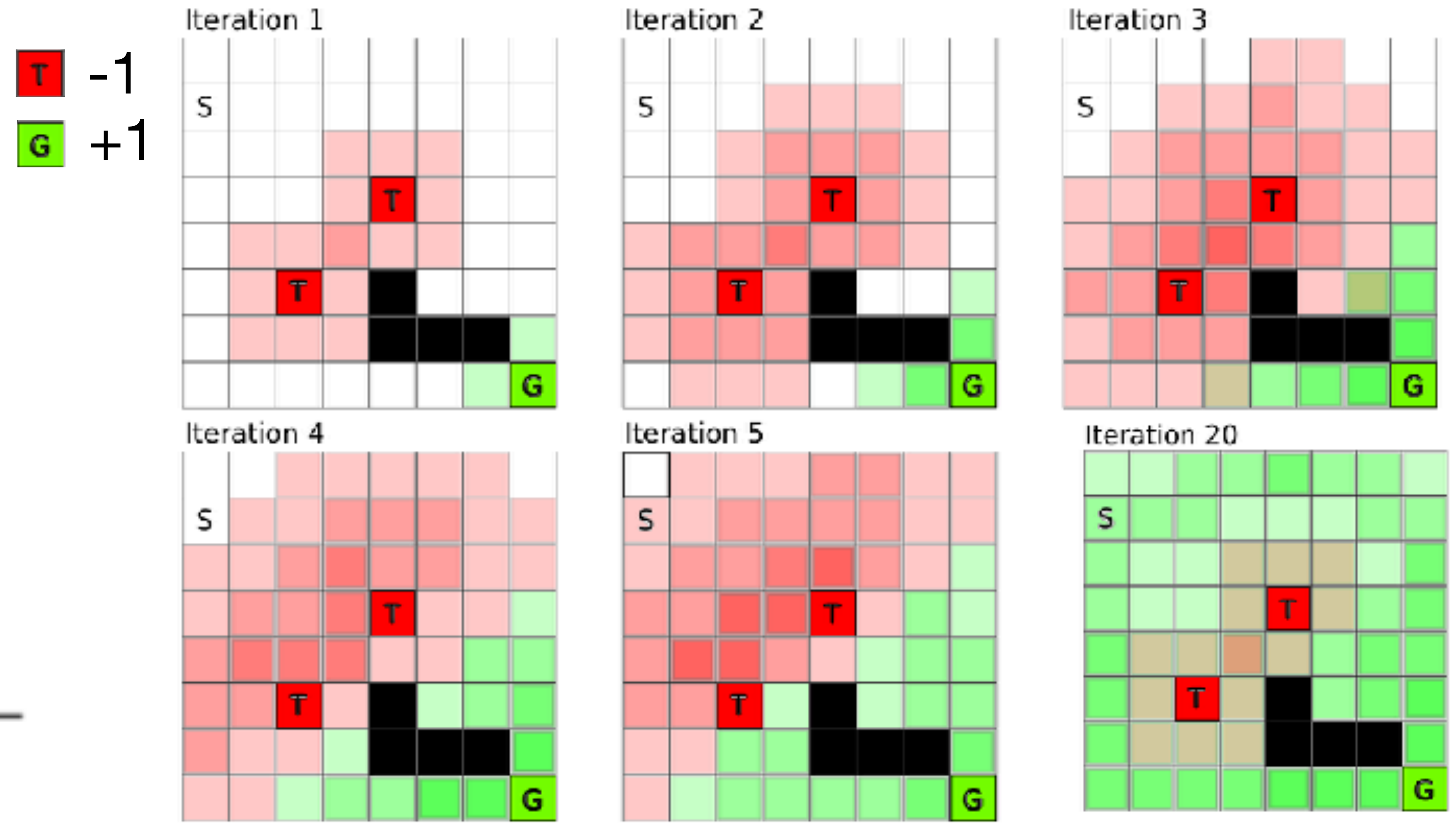

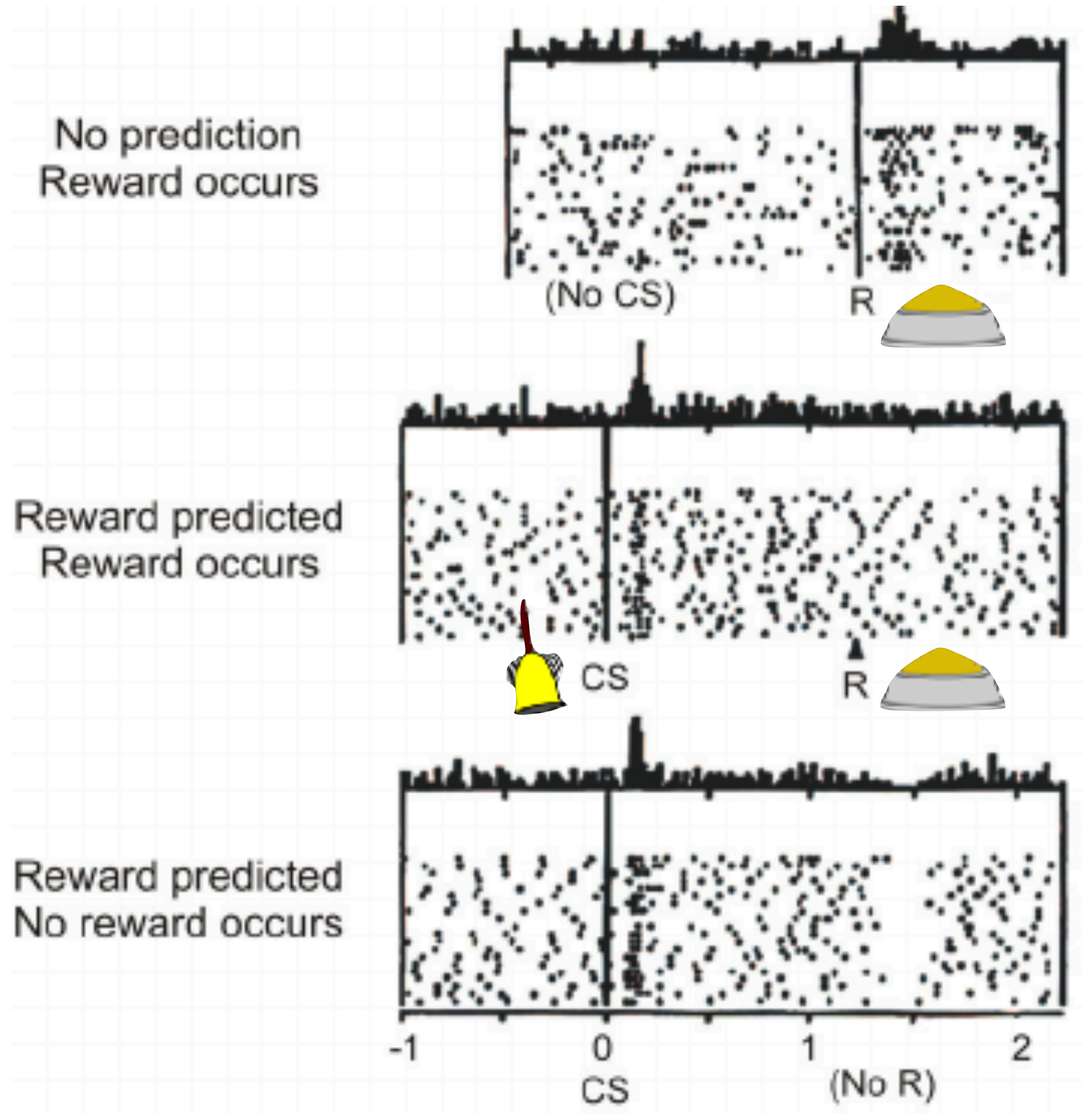

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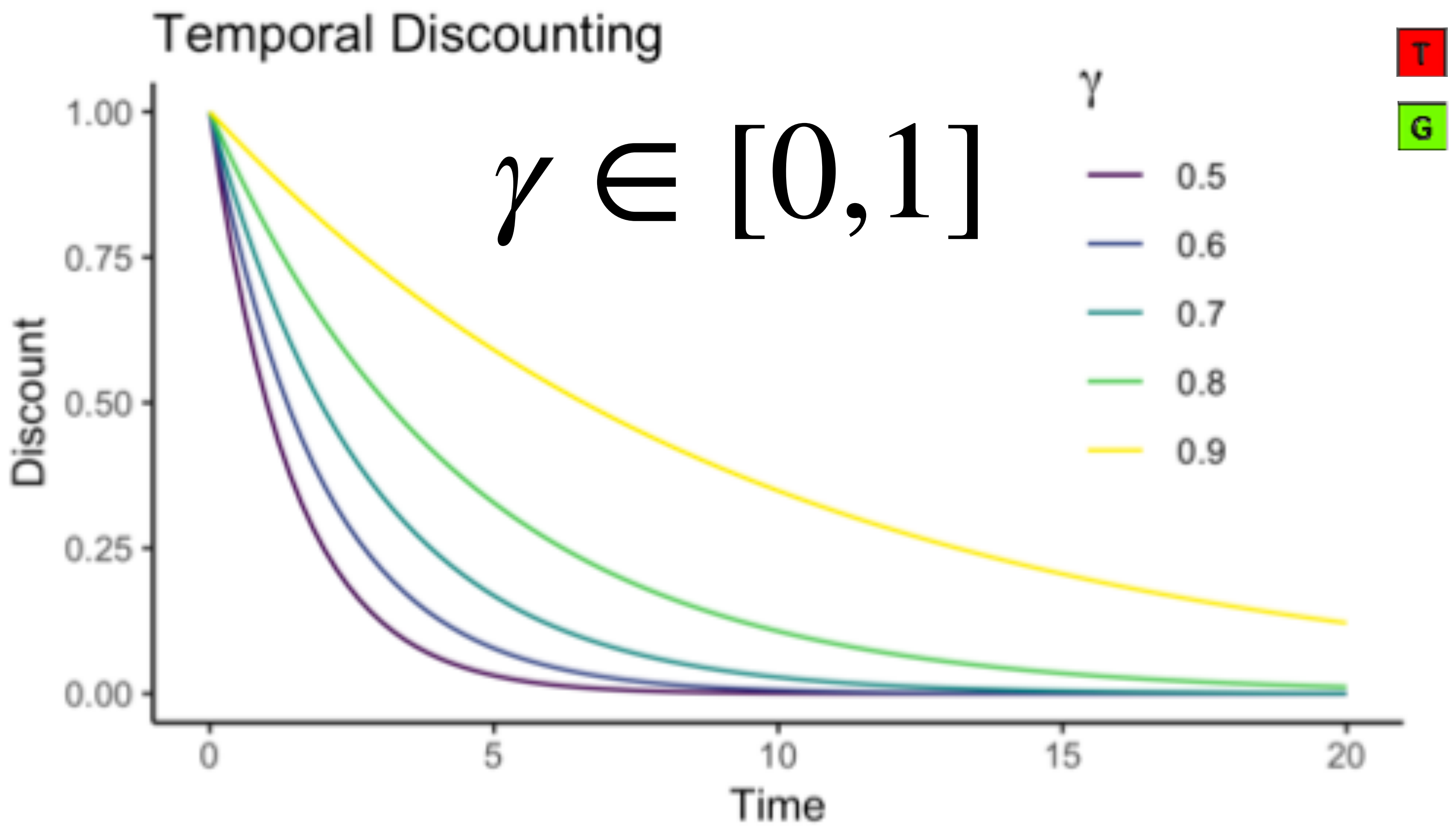

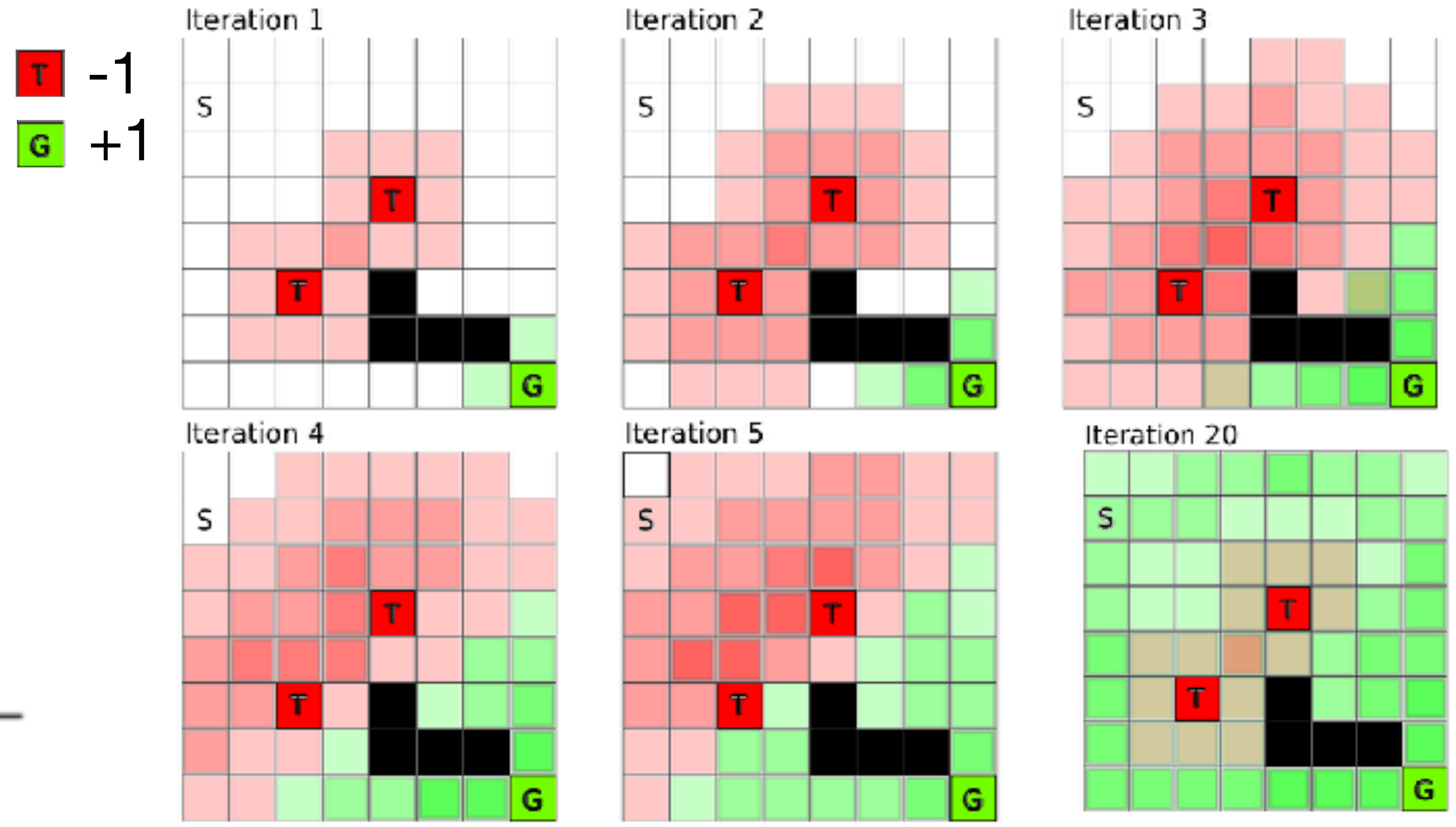

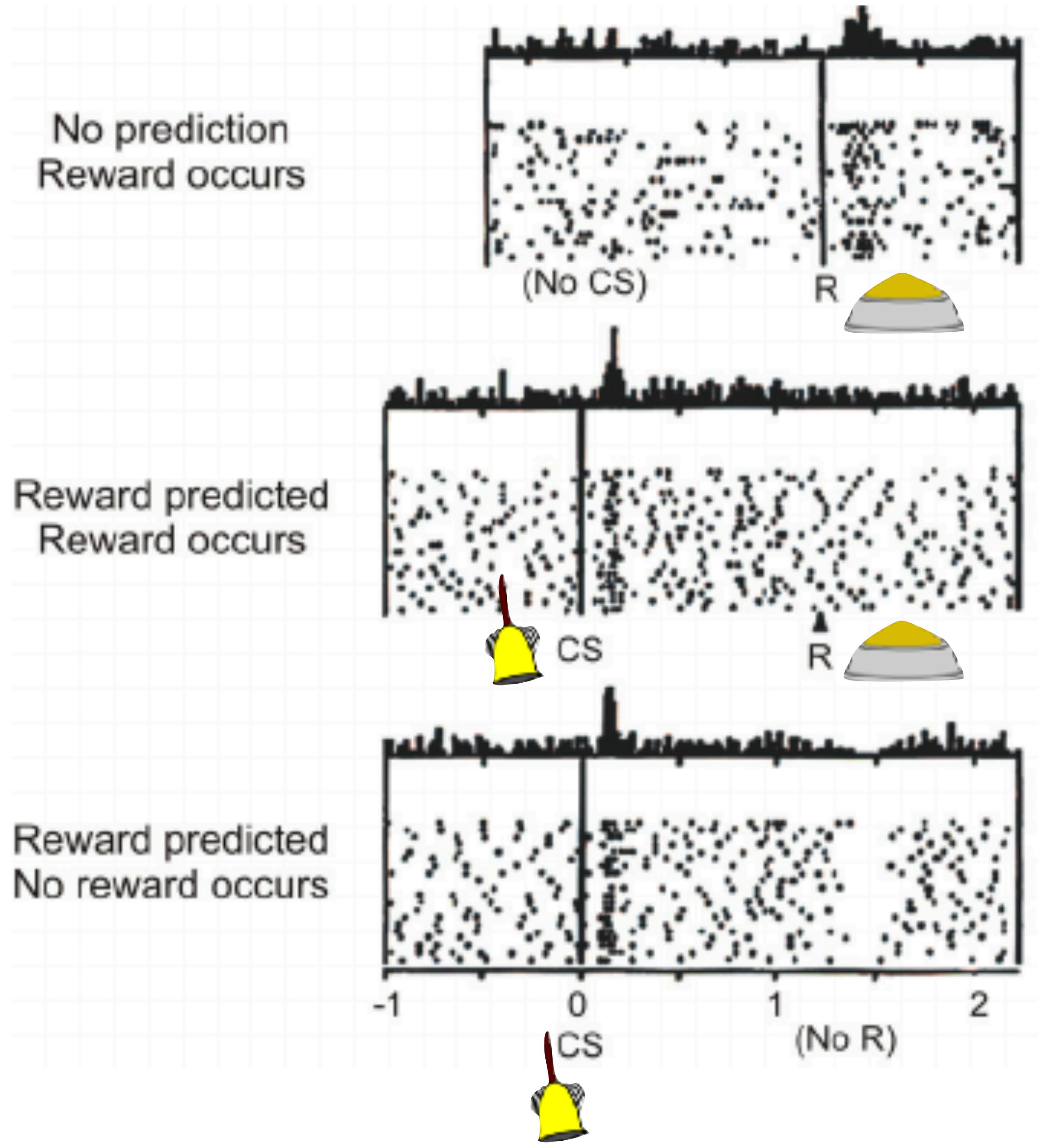

Difference between V(s) and Q(s,a)

- $V(s)$ defines how good is the state
  - Actions become implicit under policy π
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  - Actions are made explicit
- We will use $V(s)$ and $Q(s, a)$ somewhat interchangeably, depending on what the situation calls for
- It's ok to be somewhat confused at times, and I promise not to ask any "gotcha" questions purposefully trying to trick you into confusing the two

Delta-rule update with TD error

$$V(s) \leftarrow V(s) + \eta \left(r + \gamma V(s') - V(s)\right)$$

$$Q(s, a) \leftarrow Q(s, a) + \eta[r + \gamma \max_{a'} Q(s', a') - Q(s, a)]$$

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$$V_\pi(s) = \sum_a \pi(a \mid s) Q_\pi(s,a)$$

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Select a policy π^* that maximizes expected rewards

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Not just immediate rewards, but discounted future returns

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$$V_\pi(s) = \mathbb{E}_{\tau \sim \pi}[\sum_{t \in \tau} \gamma^t R_{t+1} \,|\, s_0 = s]$$

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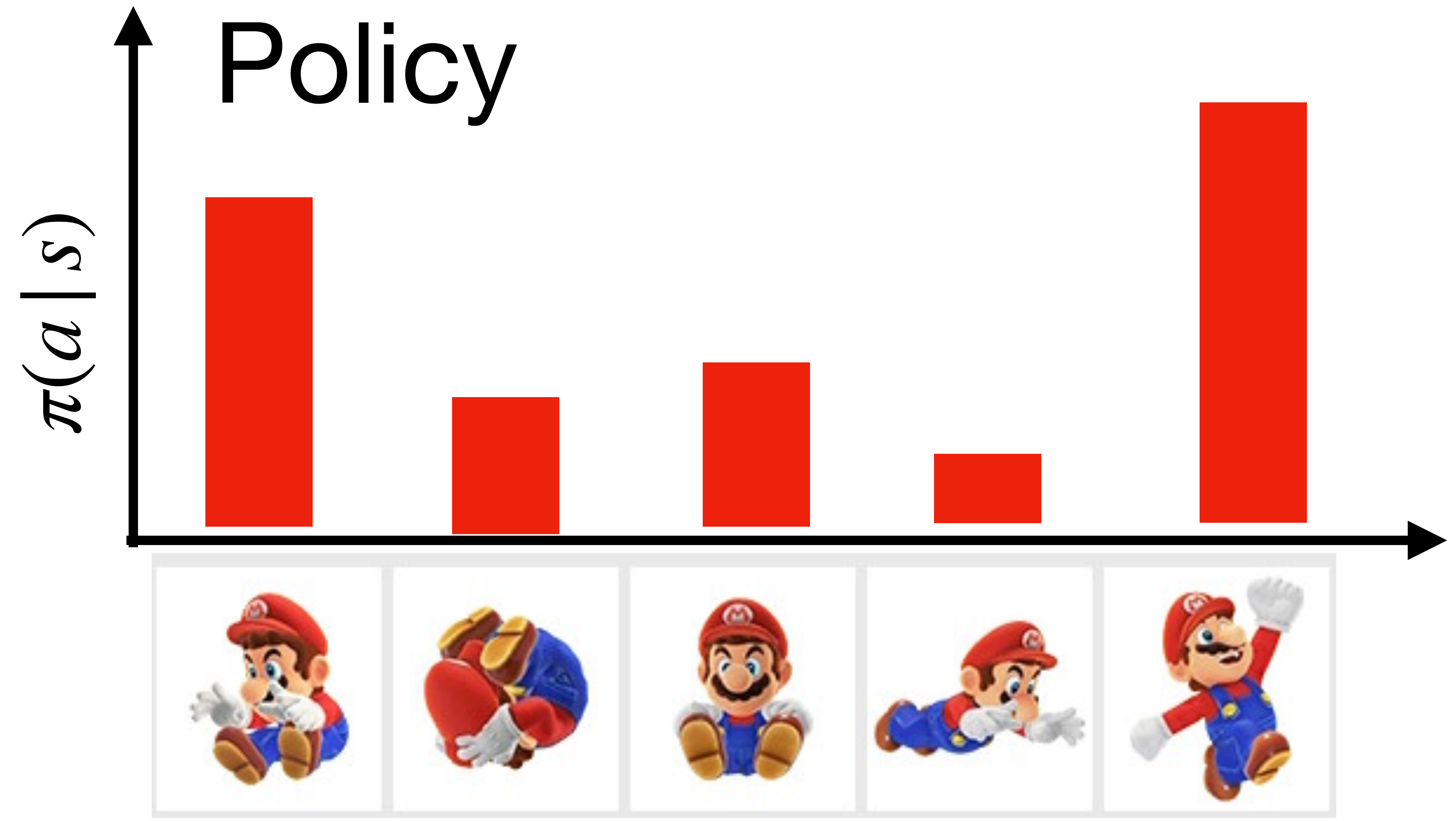
- We can rewrite the expectation $\mathbb{E}_{\tau \sim \pi}$ in terms of the policy and state transitions

$$V_\pi(s) = \sum_a \pi(a \,|\, s) \sum_{s'} P(s' \,|\, s, a)\big[R(s', a) + \gamma V_\pi(s')\big]$$

The (formal) RL Problem



Select a policy π^* that maximizes expected rewards

Not just immediate rewards, but discounted future returns

- Value function under some policy π:

$$V_\pi(s) = \mathbb{E}_{\tau \sim \pi}[\sum_{t \in \tau} \gamma^t R_{t+1} \mid s_0 = s]$$

- We can rewrite the expectation $\mathbb{E}_{\tau \sim \pi}$ in terms of the policy and state transitions

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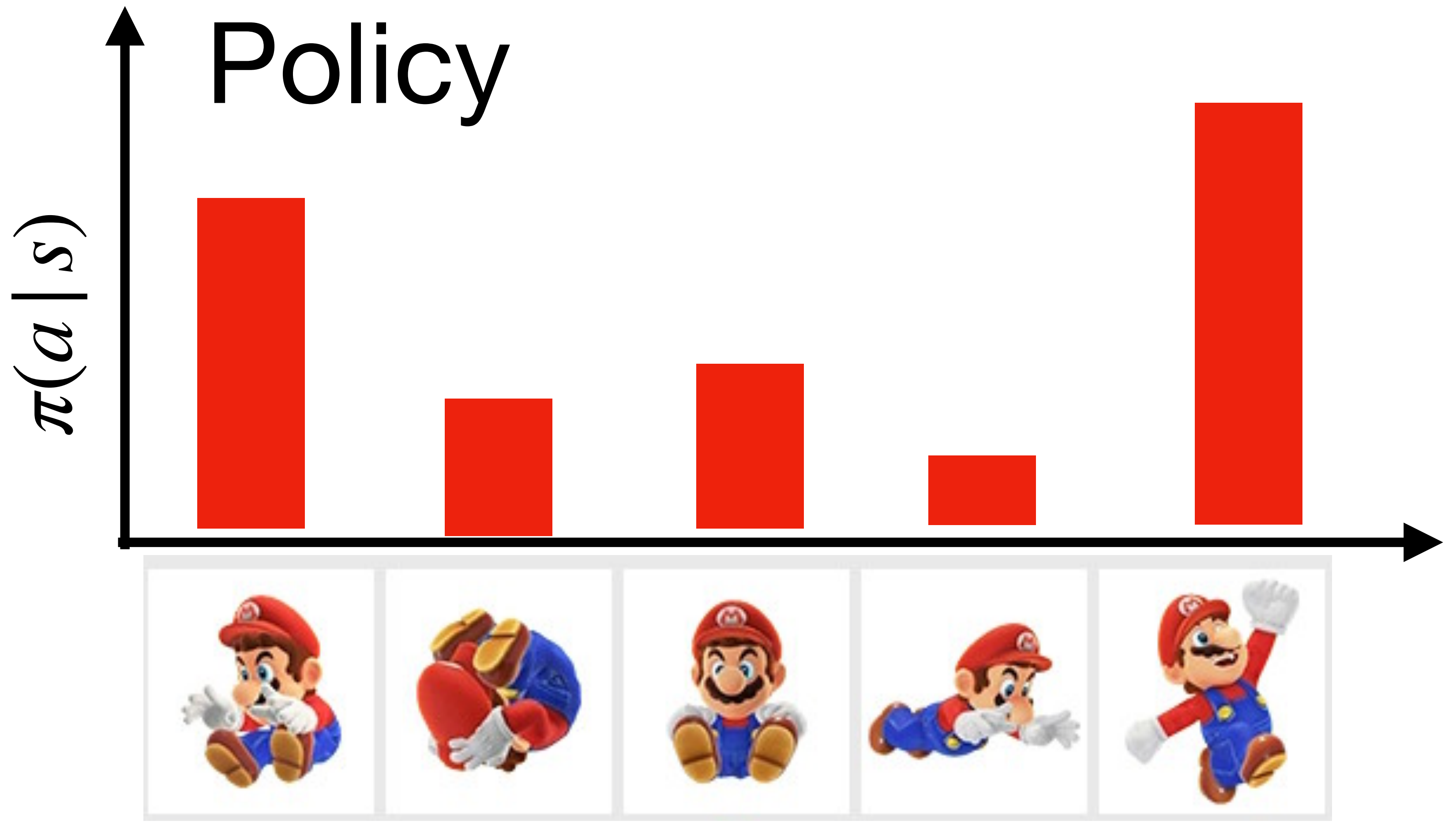

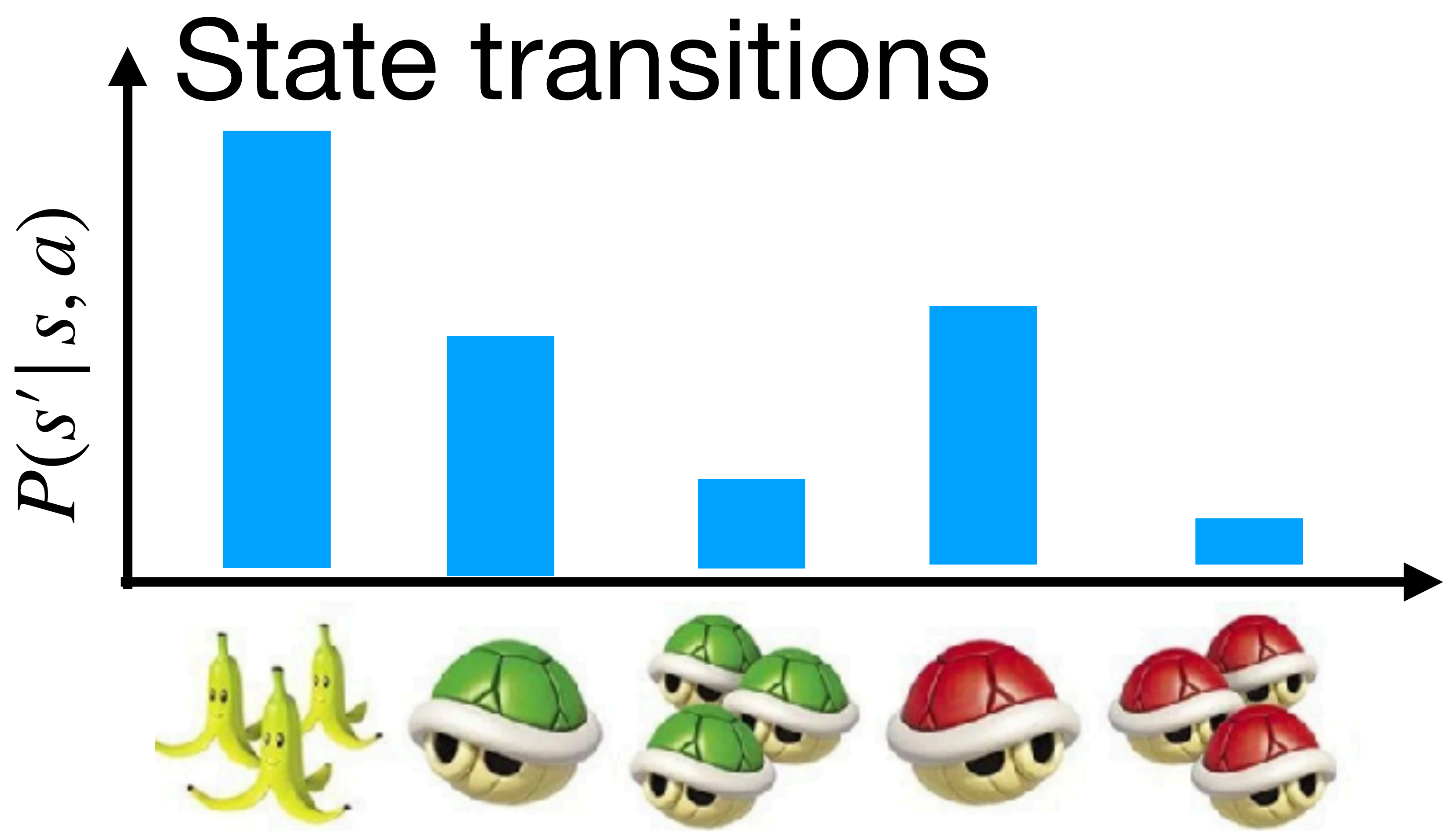

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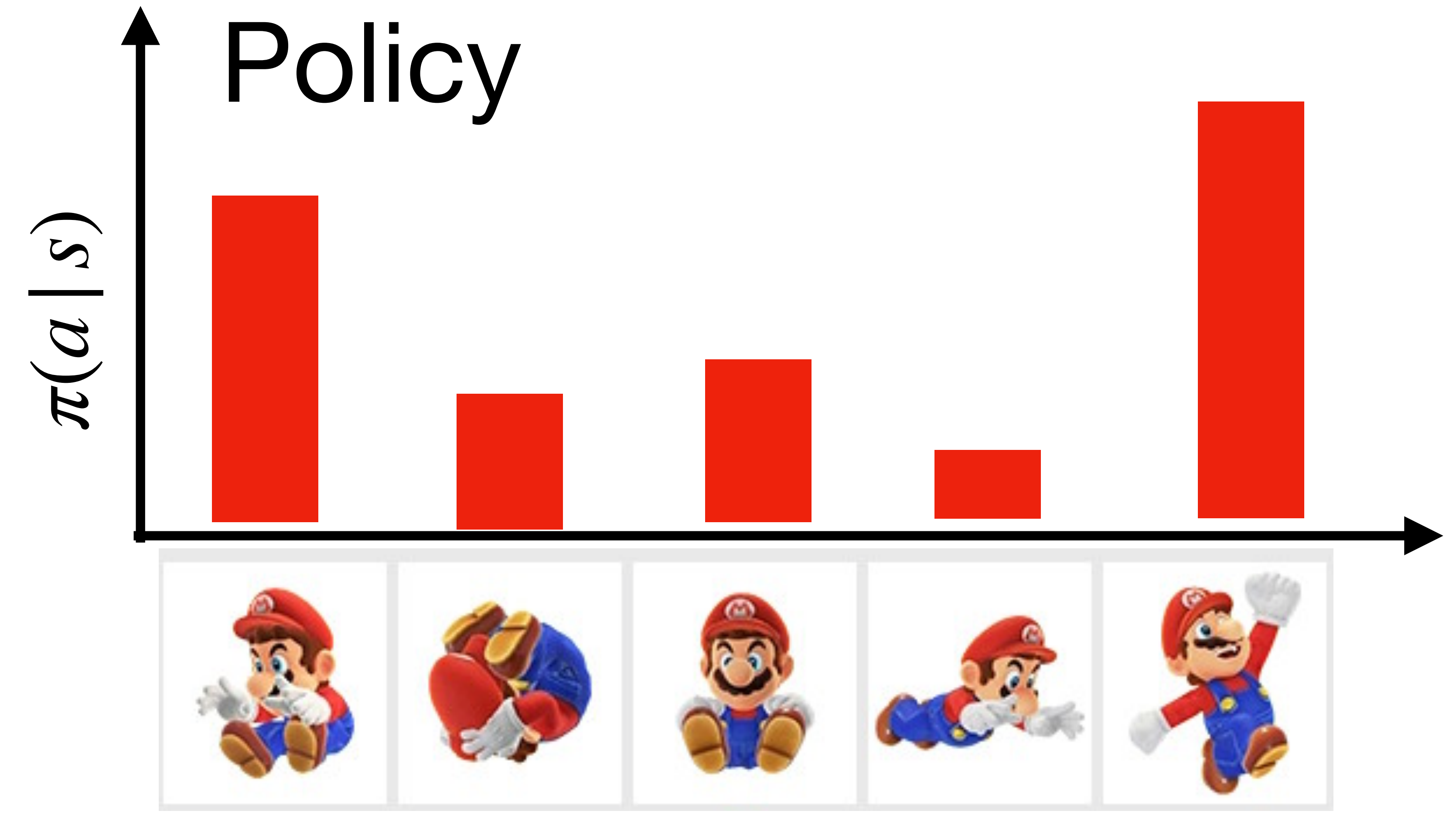

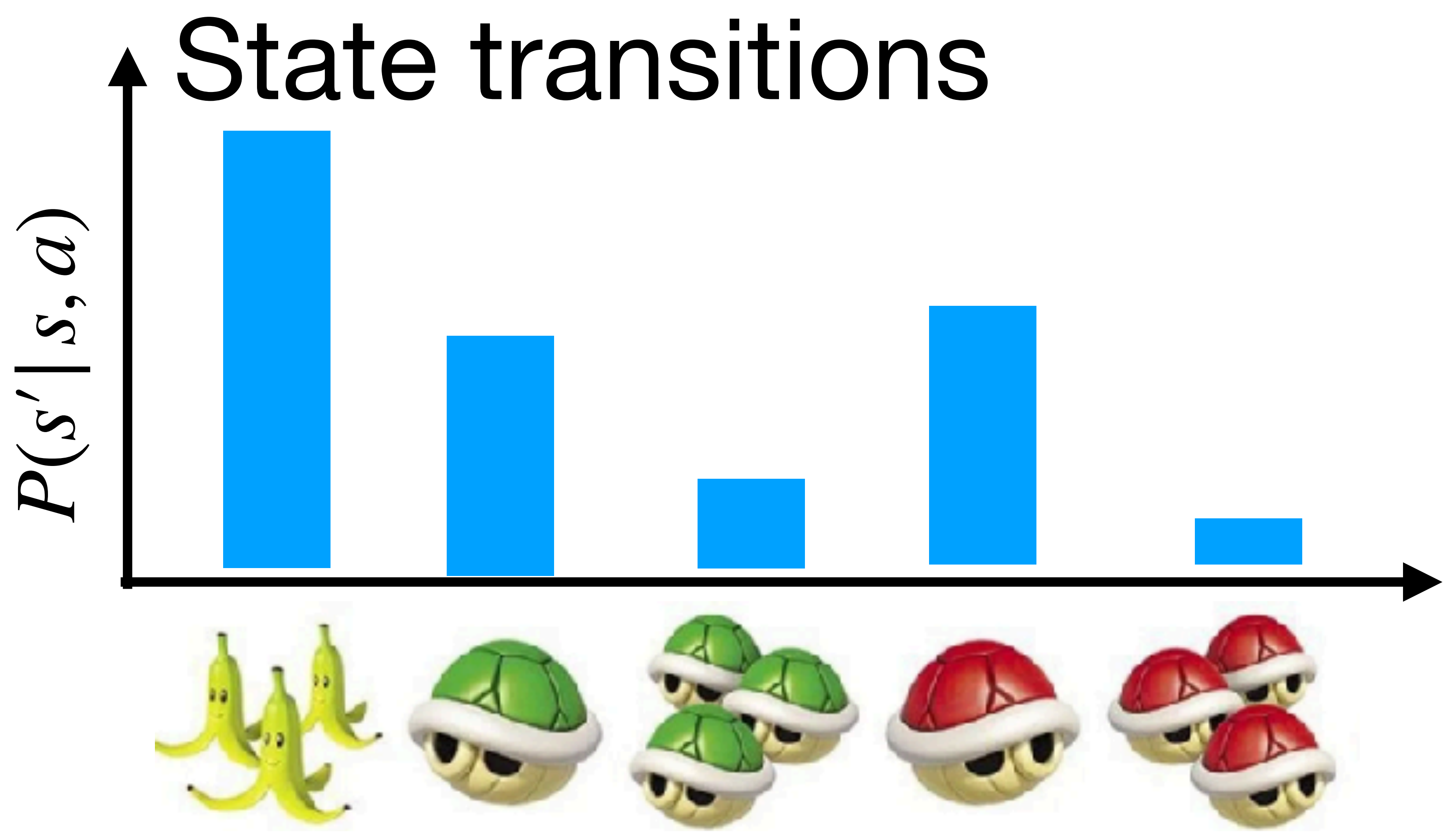

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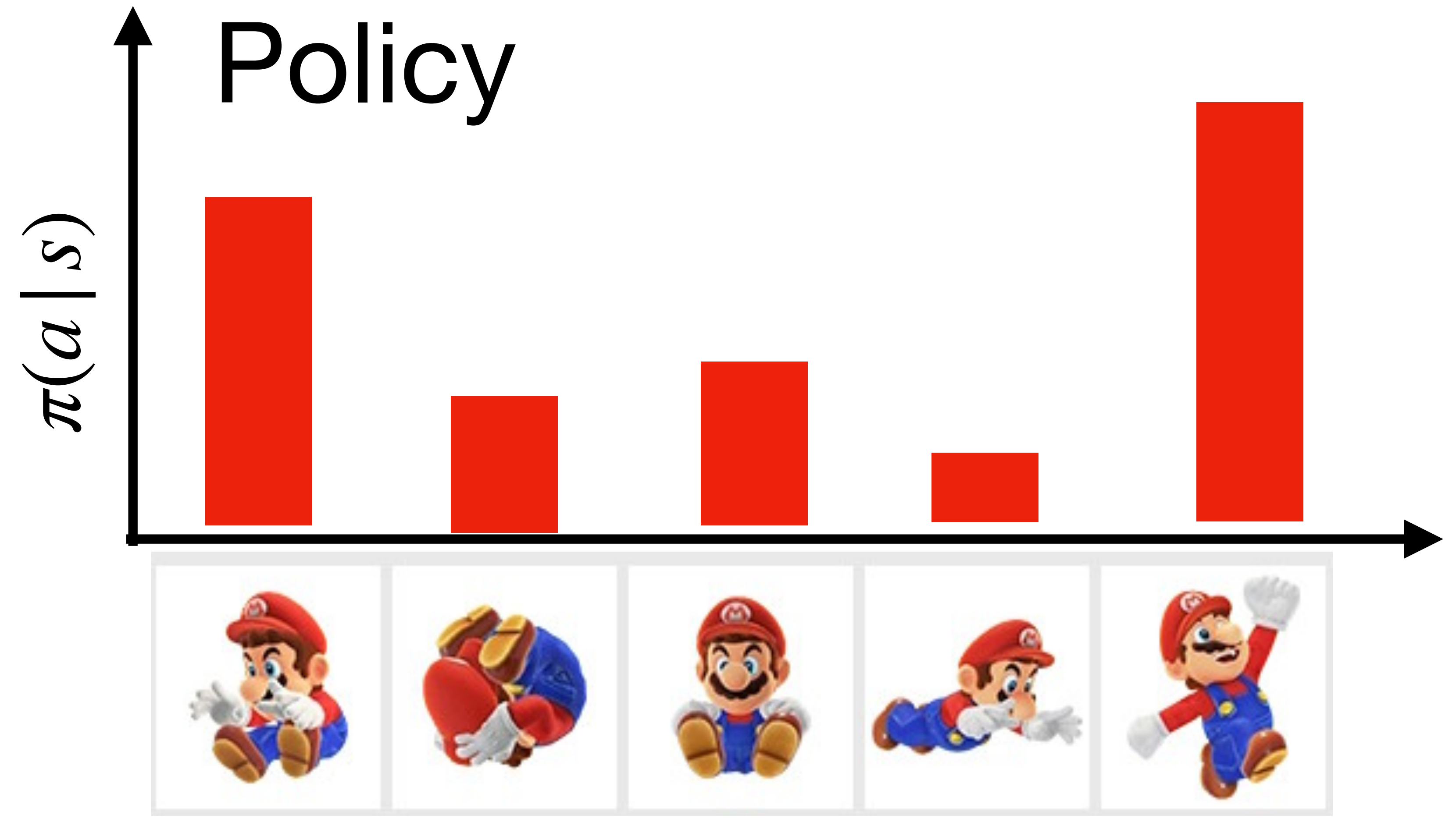

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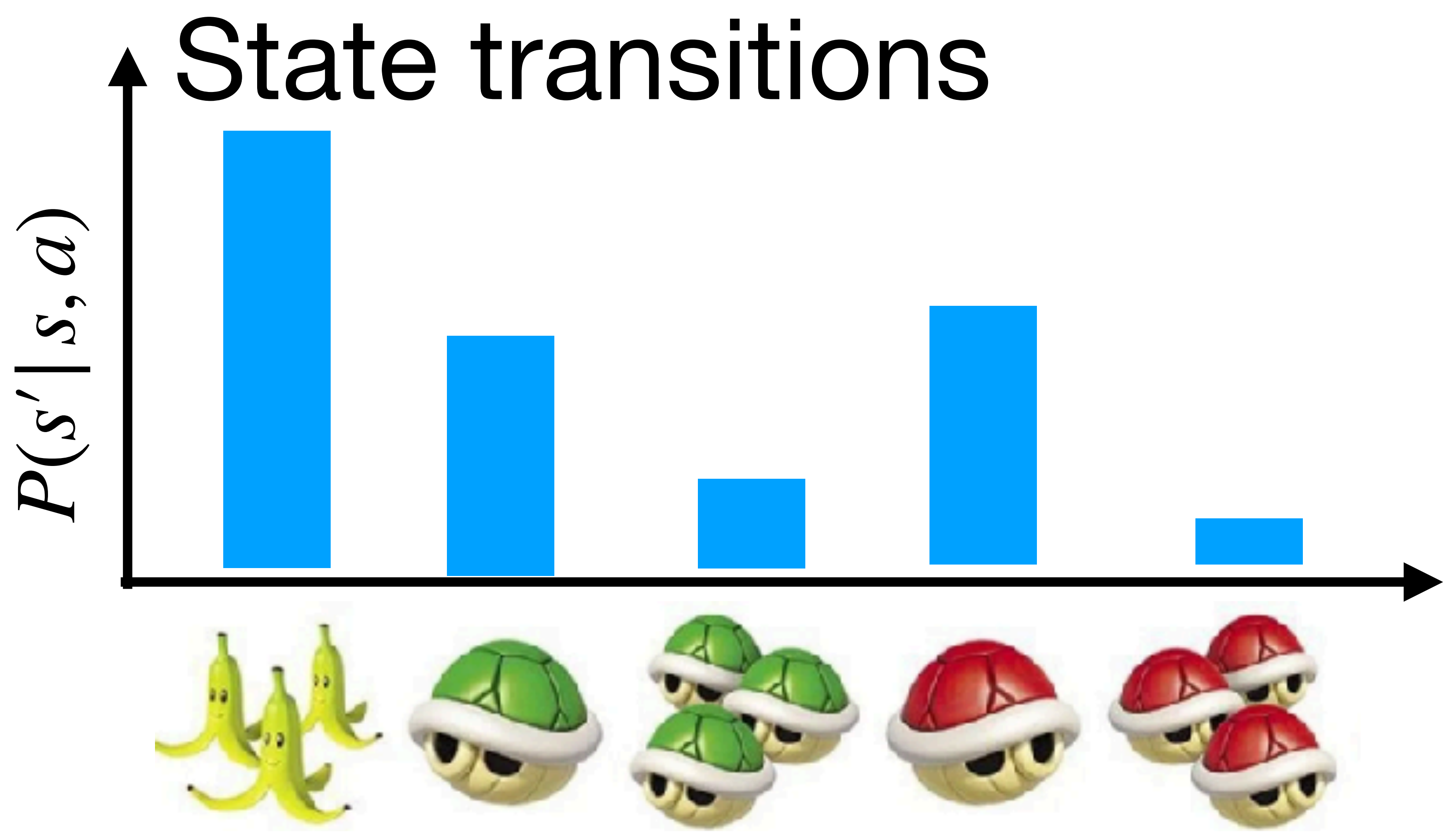

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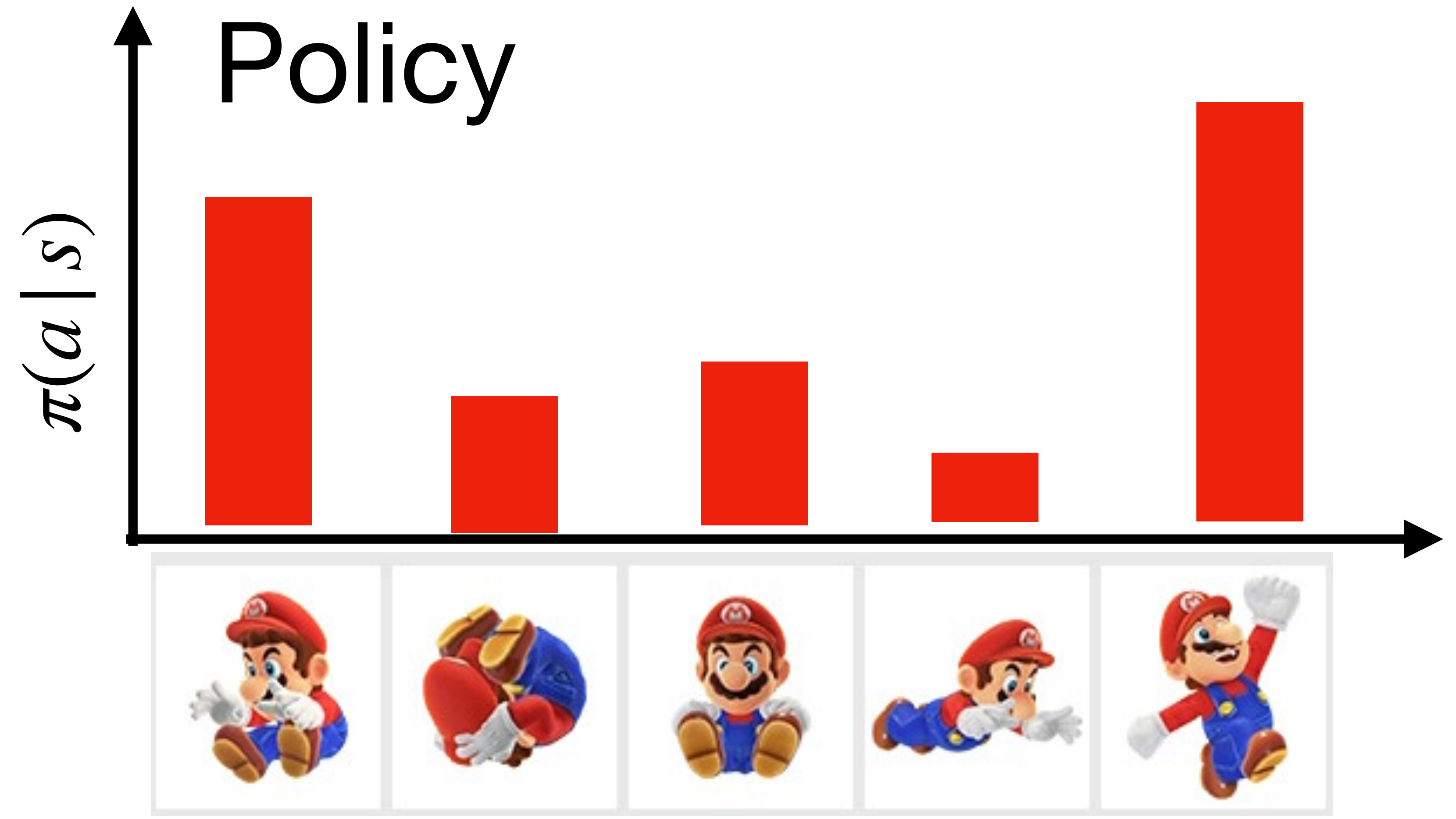

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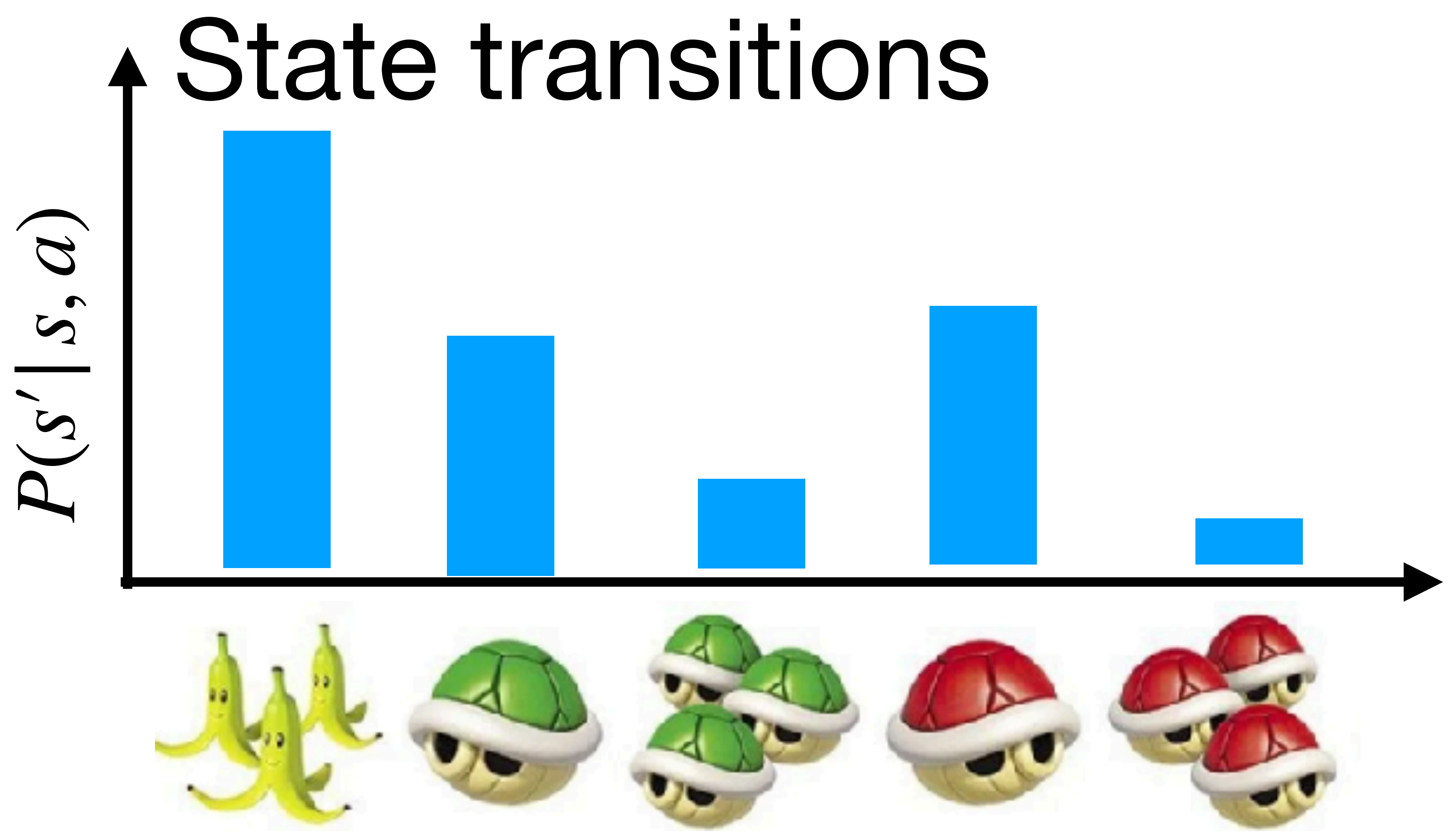

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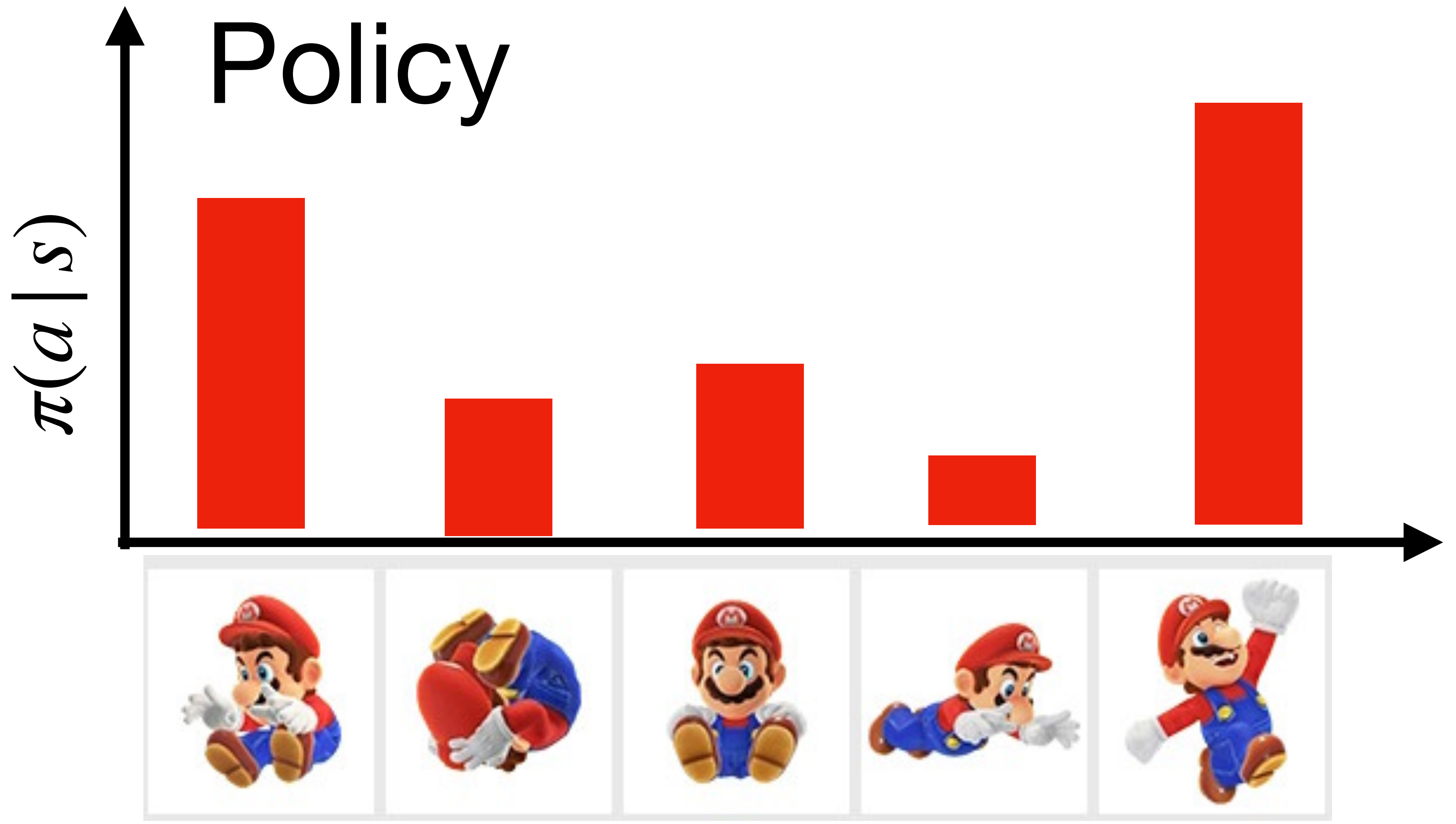

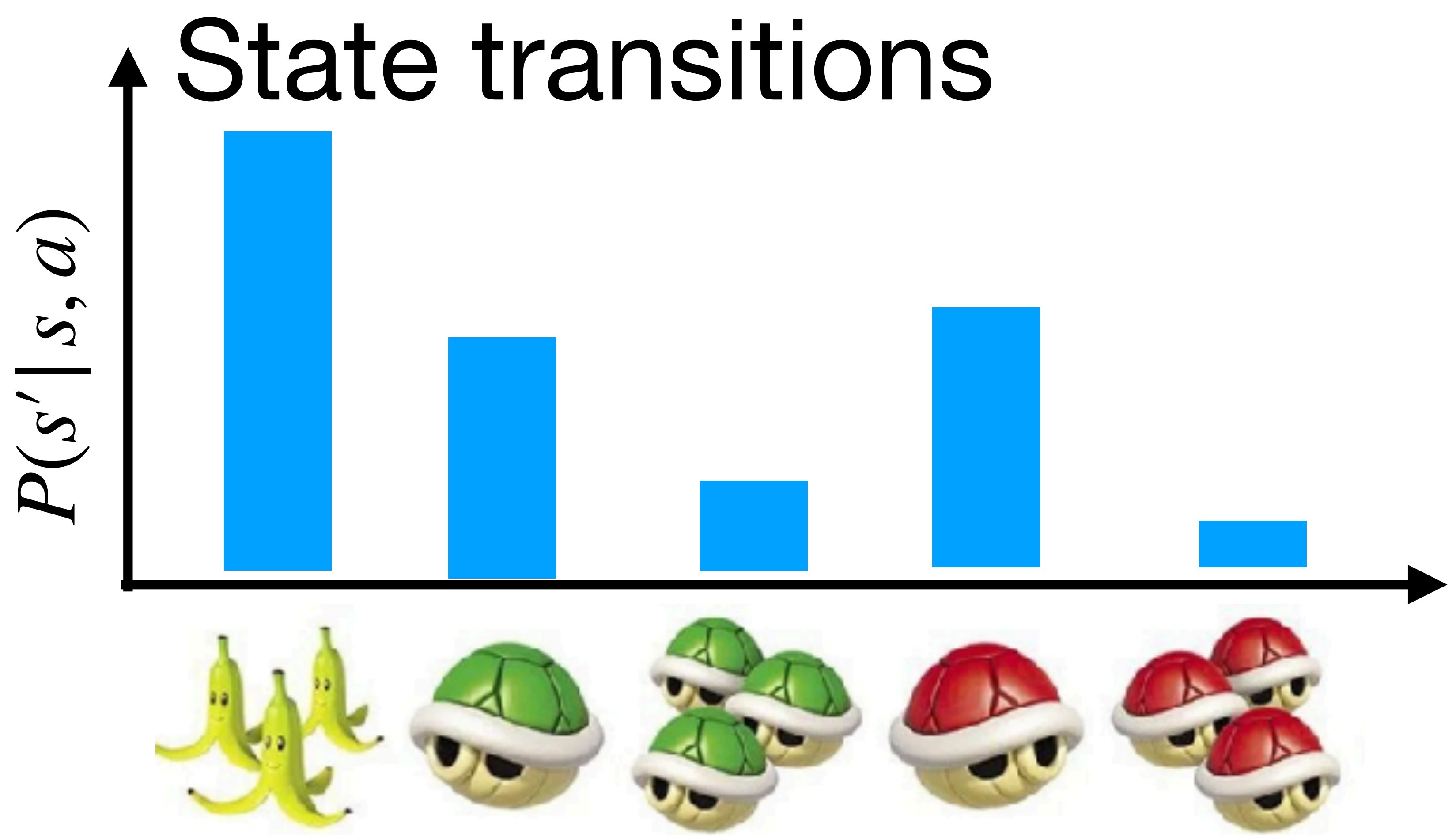

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Optimal policies via Bellman Equations

Memorizing these equations not neccessary for exam

- This recursive formulation of the value function is known as **the Bellman equation**

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  - This allows us to break the optimization problem into series of simpler sub-problems
    - if each sub-problem is solved optimally, the overall problem will also be optimal
- Note that there is no longer any **reward prediction error** updating
- Rather, we want to describe a **theoretically optimal solution**:
  - We first define an optimal value function by assuming value-maximizing actions:

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  - We then (theoretically) arrive at an optimal policy by selecting actions that maximize value:

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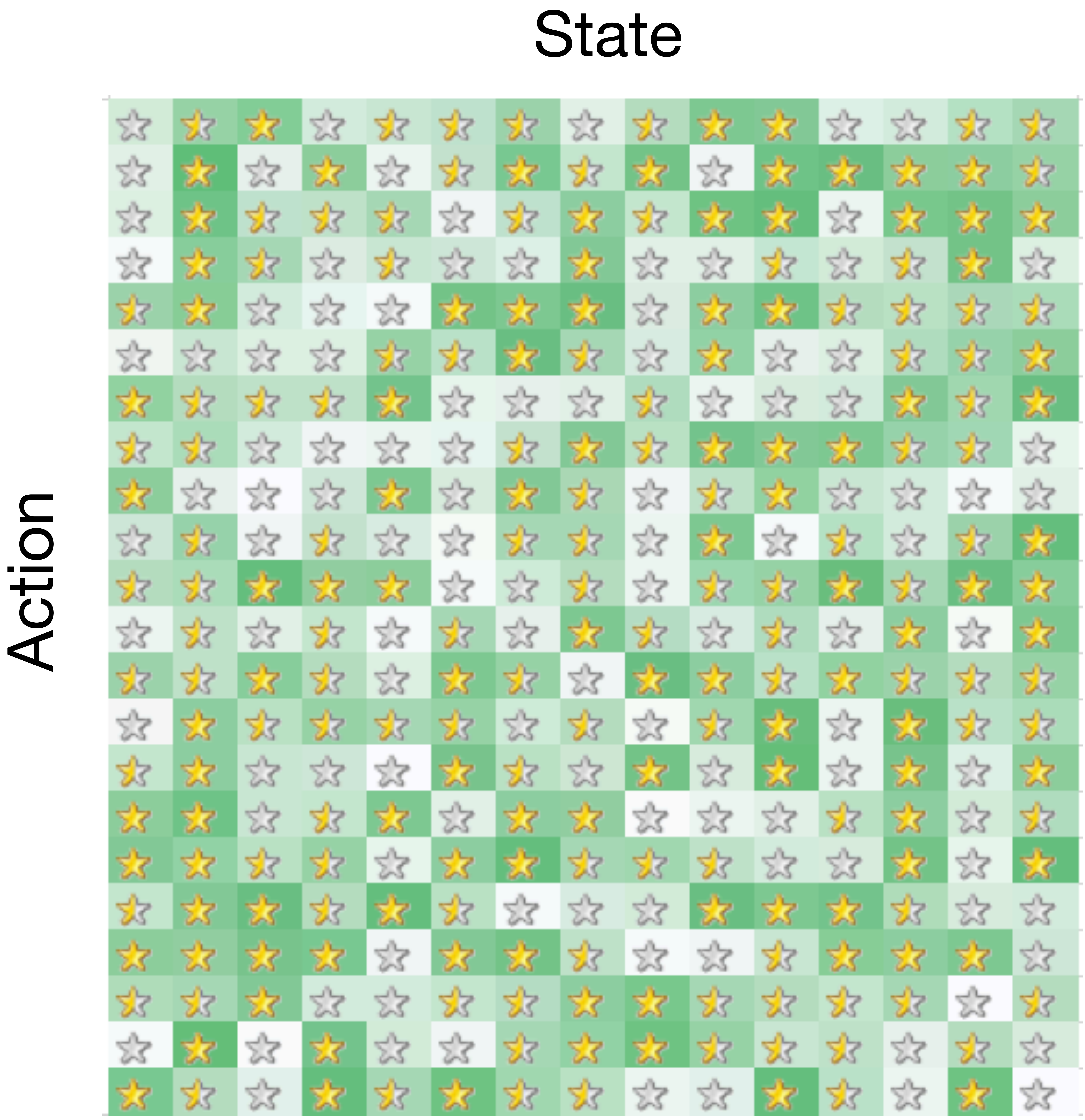
* In practice, optimal solutions are usually unobtainable

Tabular methods

- Based on methods from Dynamic programming (Bellman, 1957), Tabular methods were first proposed as solutions for RL problems by Minsky (1961)
- Think of a giant lookup table, where we store a value representation for each combination of state+action
- **Value iteration** and **policy iteration** are examples of tabular methods
- Caveat: solutions require repeat visits to each state, which is infeasible in most real-world problems



Value iteration

Iteratively visit all states and update the value function until a "good enough" solution has been reached.

1. Initialize the value function as $V_{k=0}(s) = 0$ for all states

2. For k in (1, 2, ….) update all s in $\mathcal{S}$:

$$V_{k+1}(s) = \max_{a \in A} \sum_{s'} P(s' \mid s, a)[R(s, a) + \gamma V_k(s')]$$

until $\max_{s \in \mathcal{S}} |V_k(s) - V_{k-1}(s)| < \theta$ Bellman residual

V_k converges on V_* as $k \to \infty$, and perhaps sooner, but with many costly sweeps through the state space

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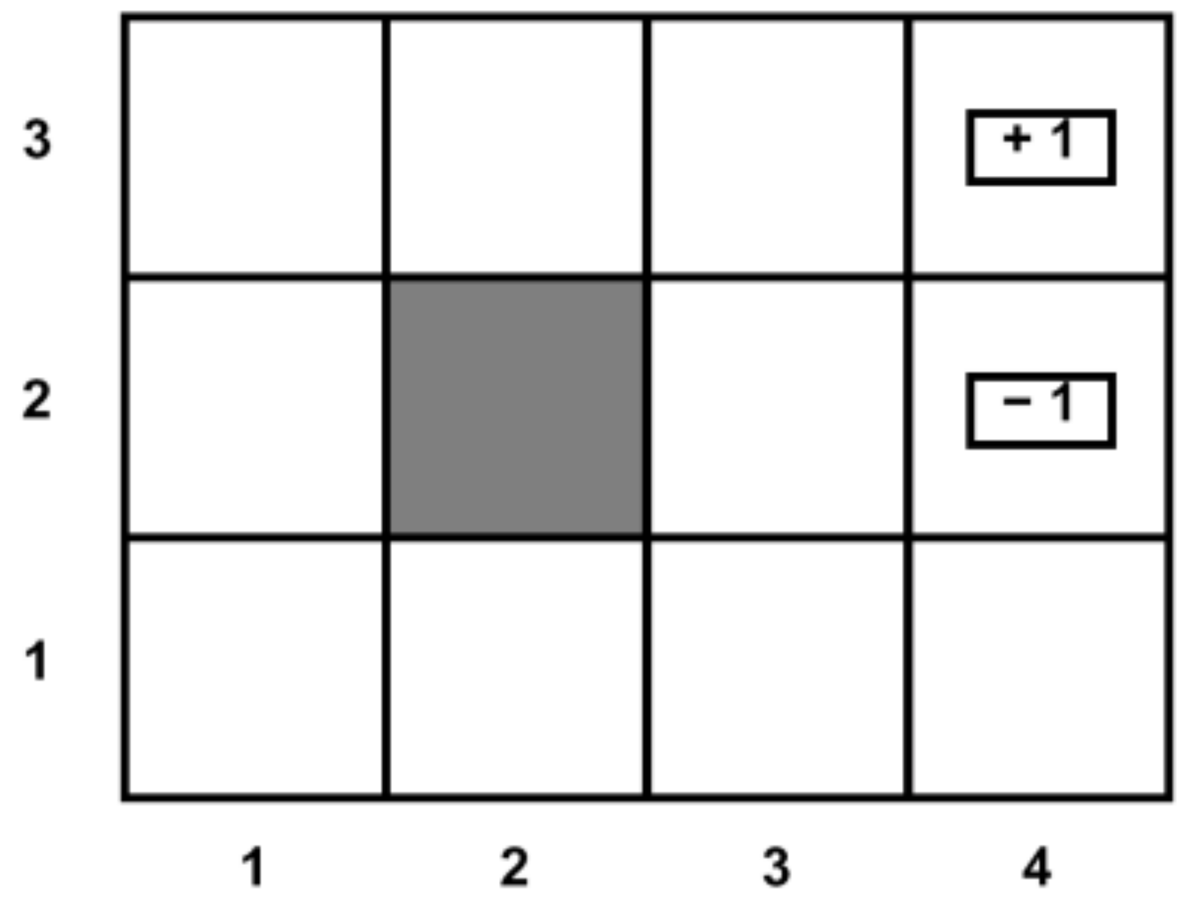

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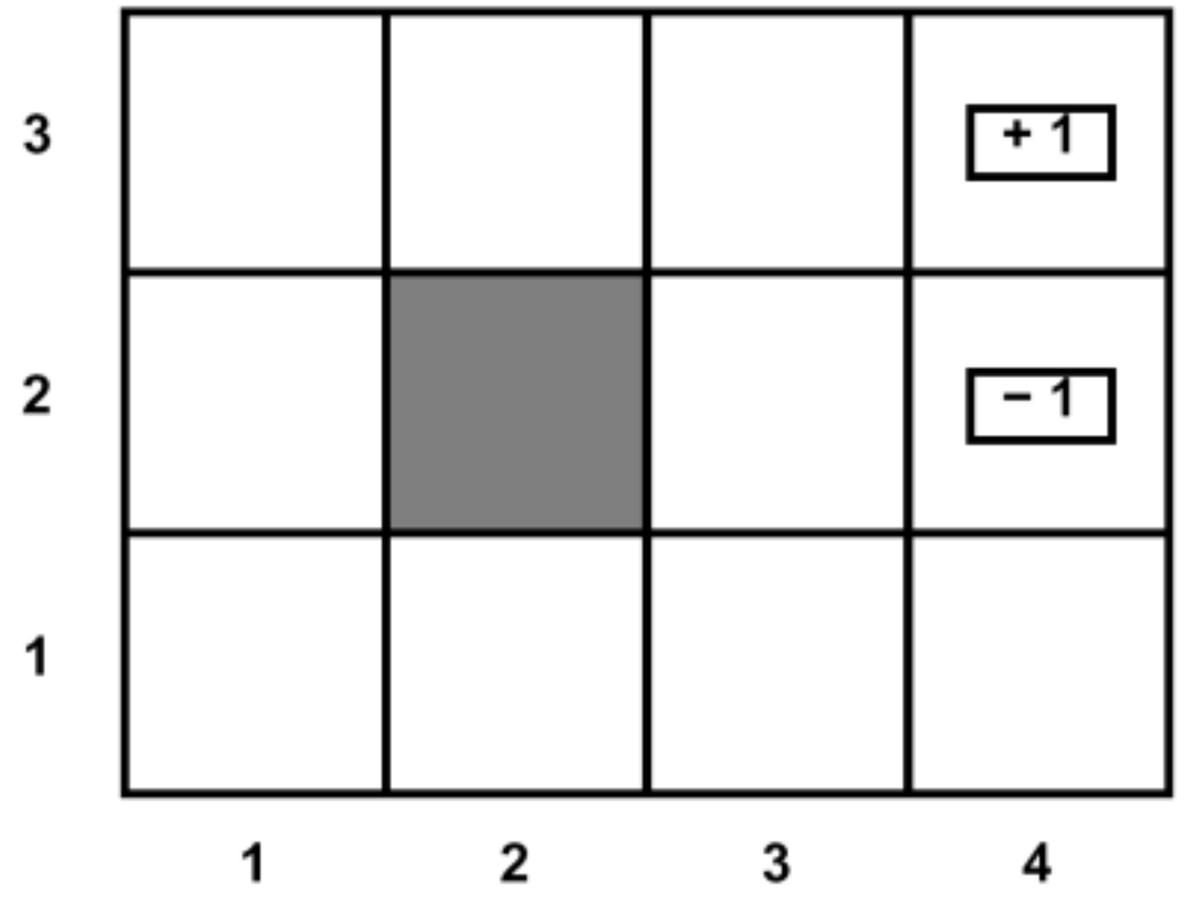

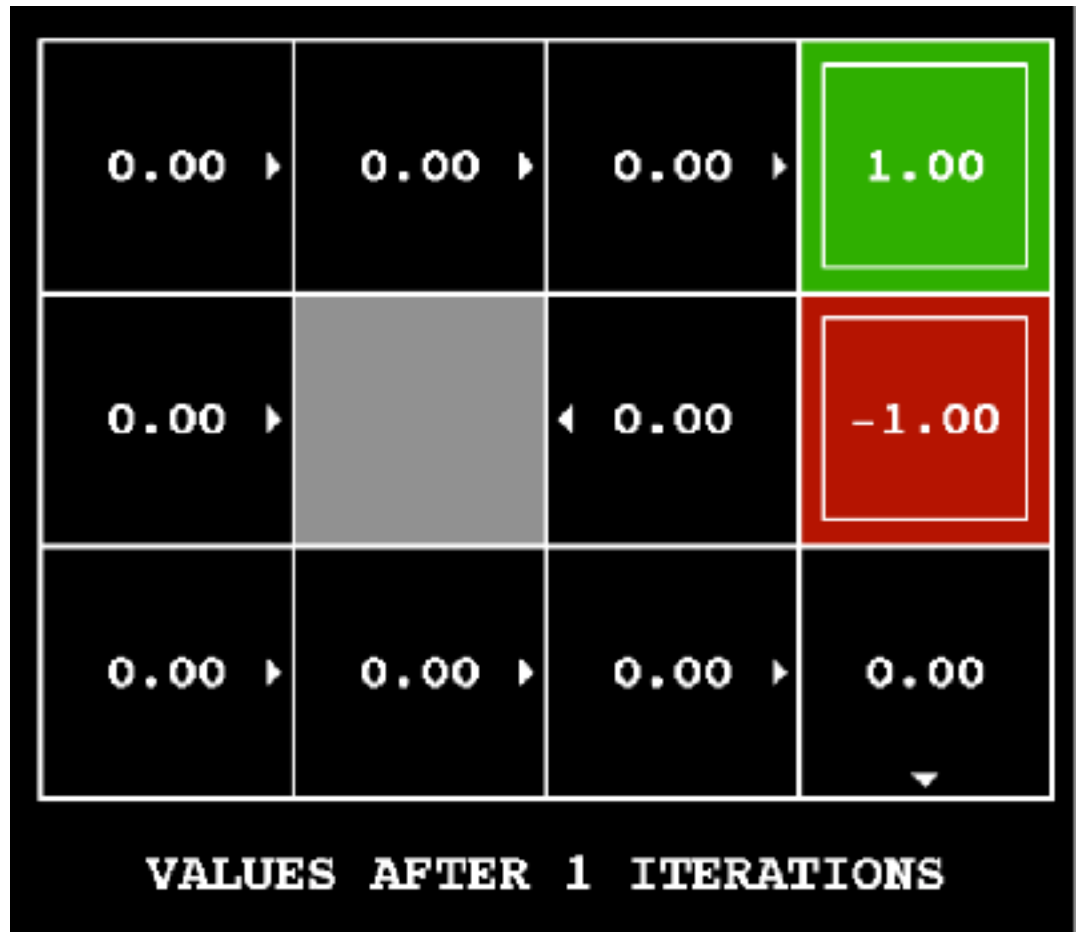

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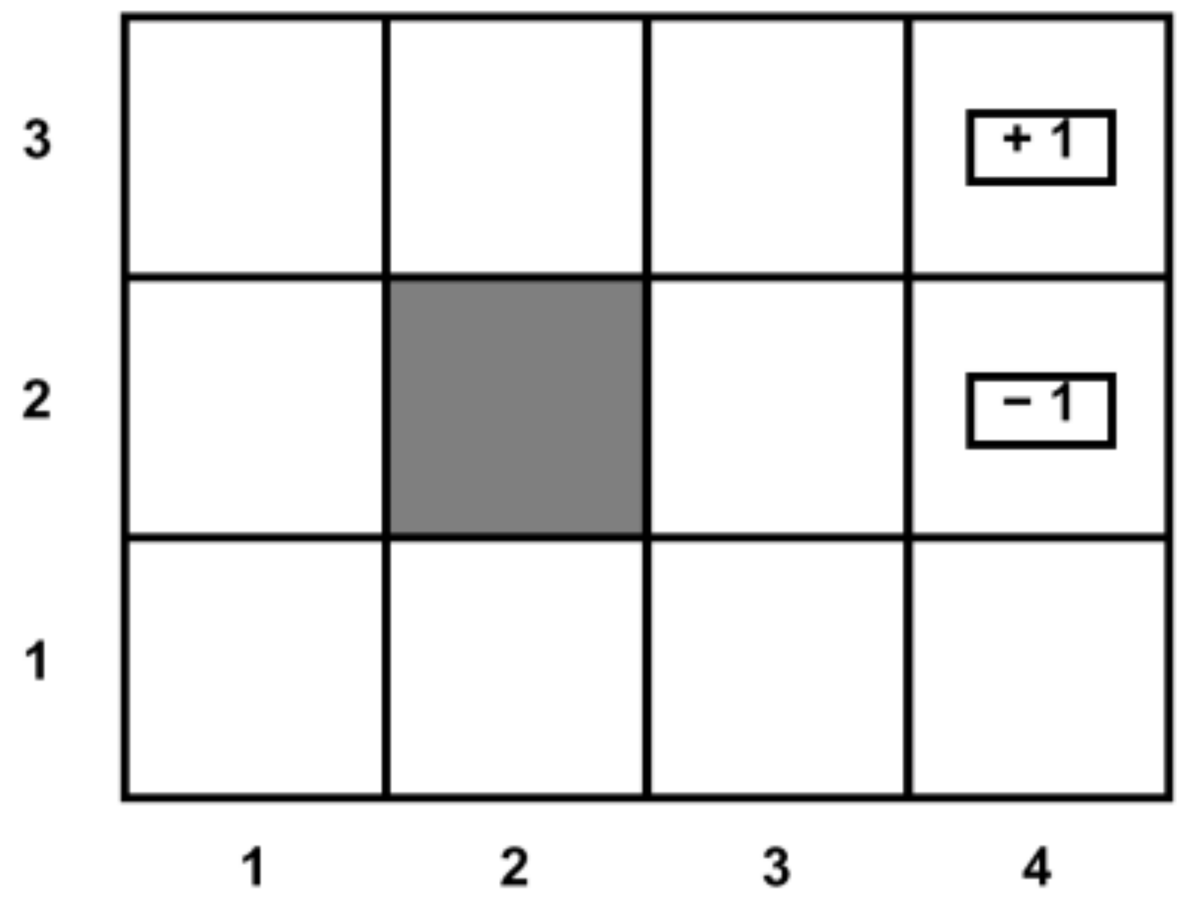

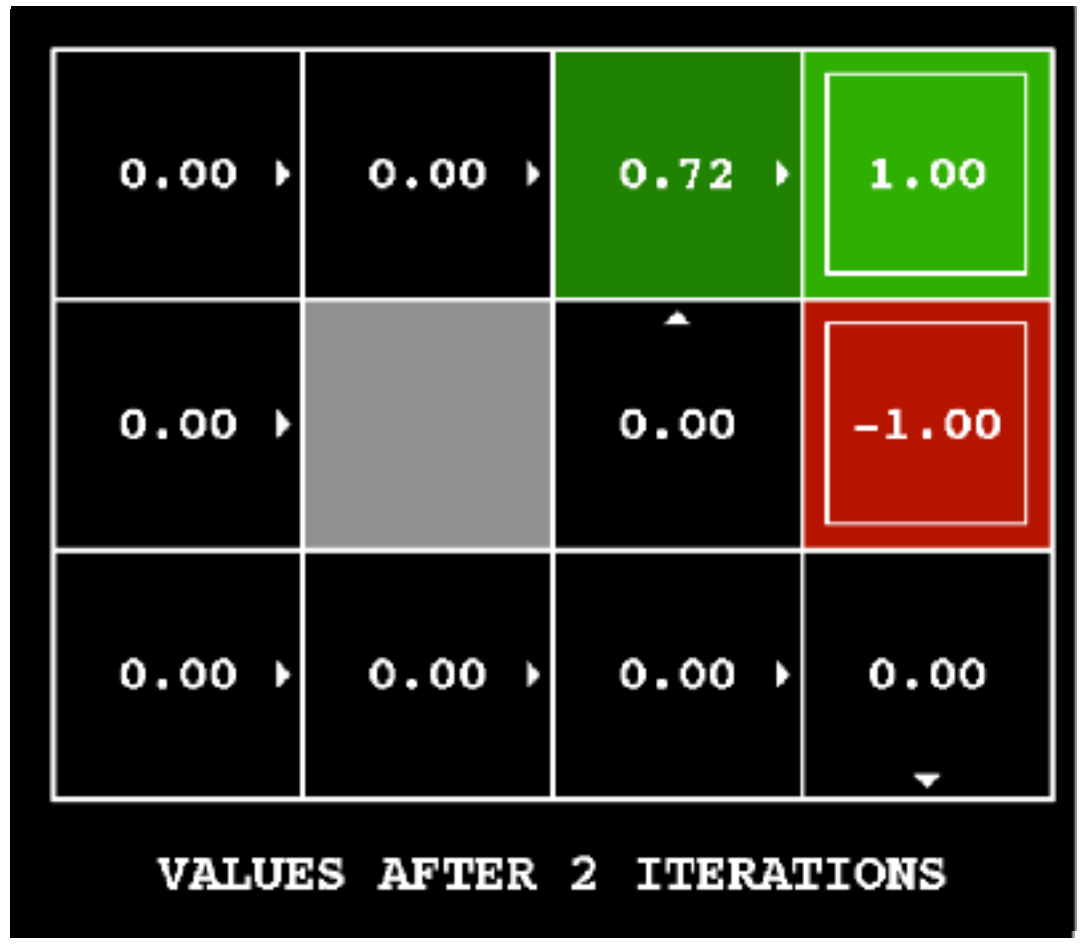

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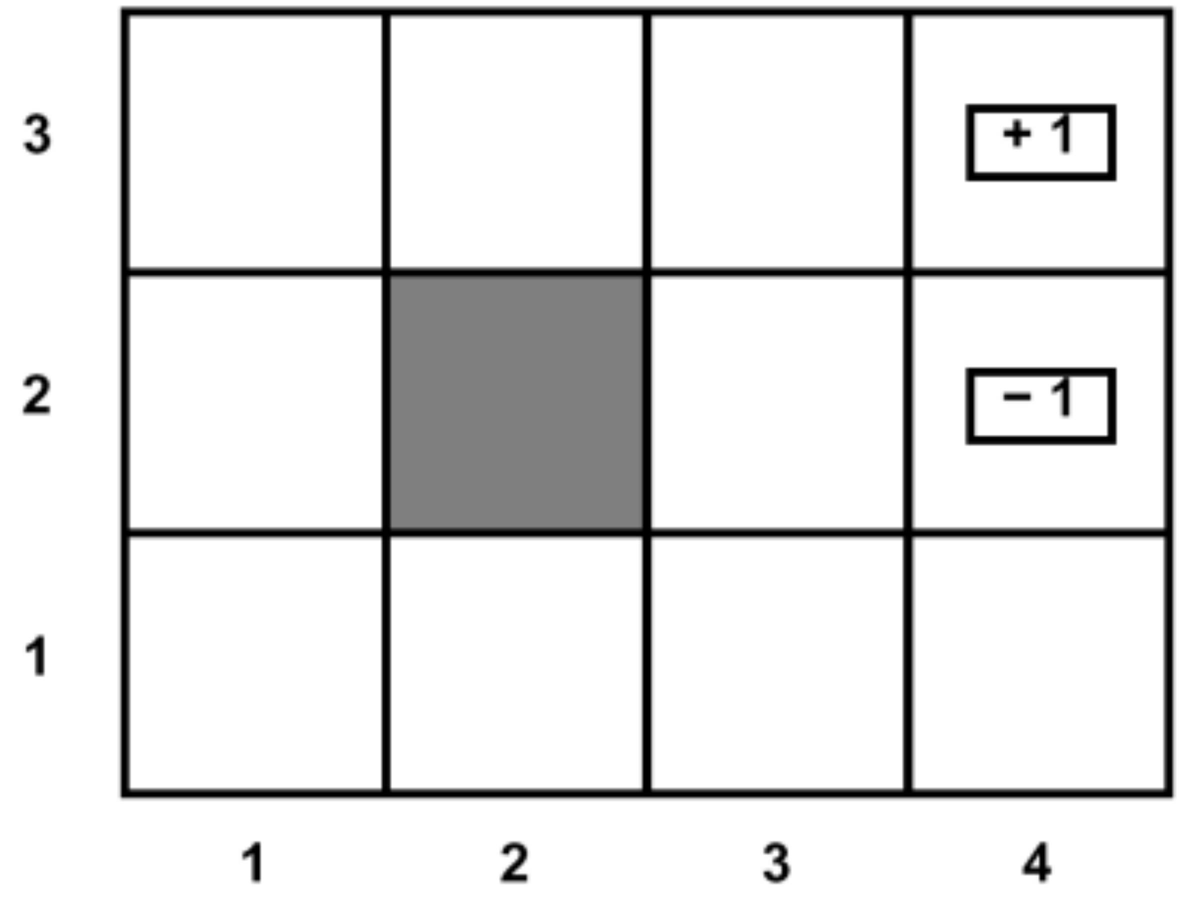

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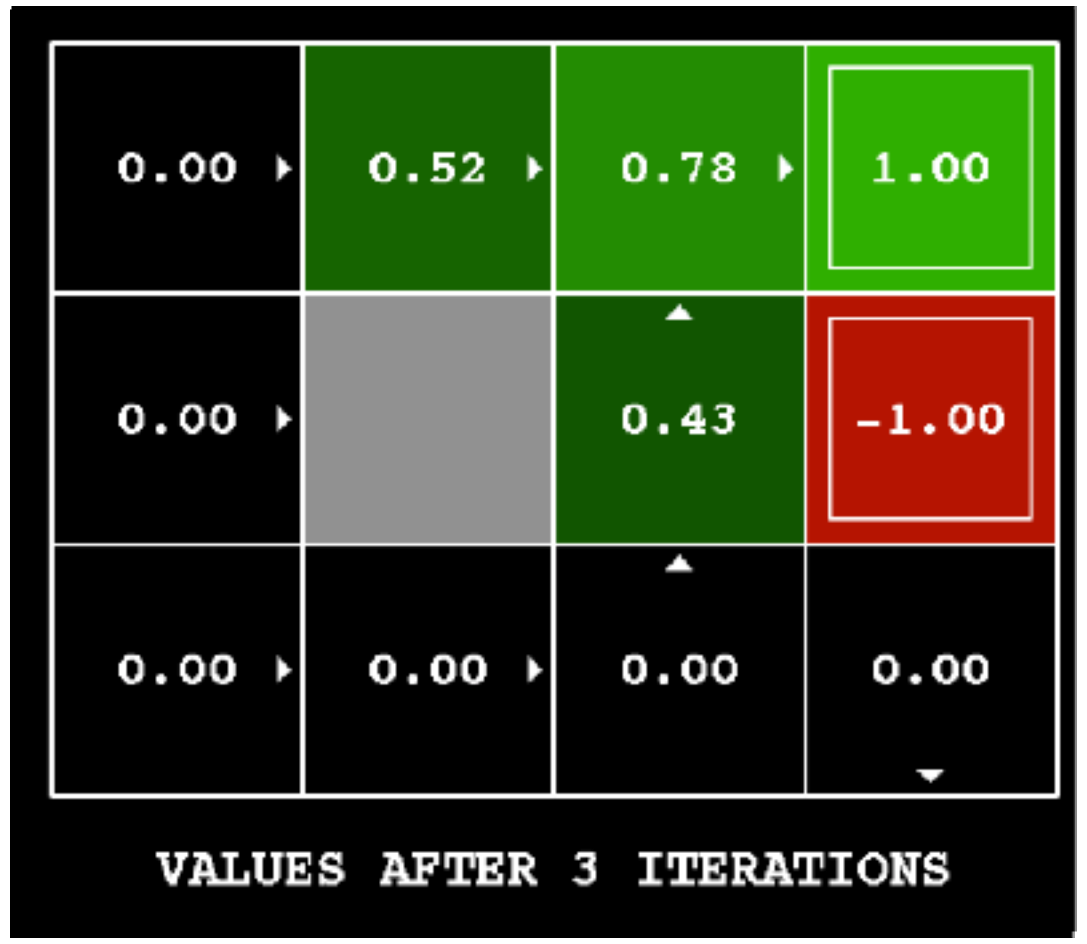

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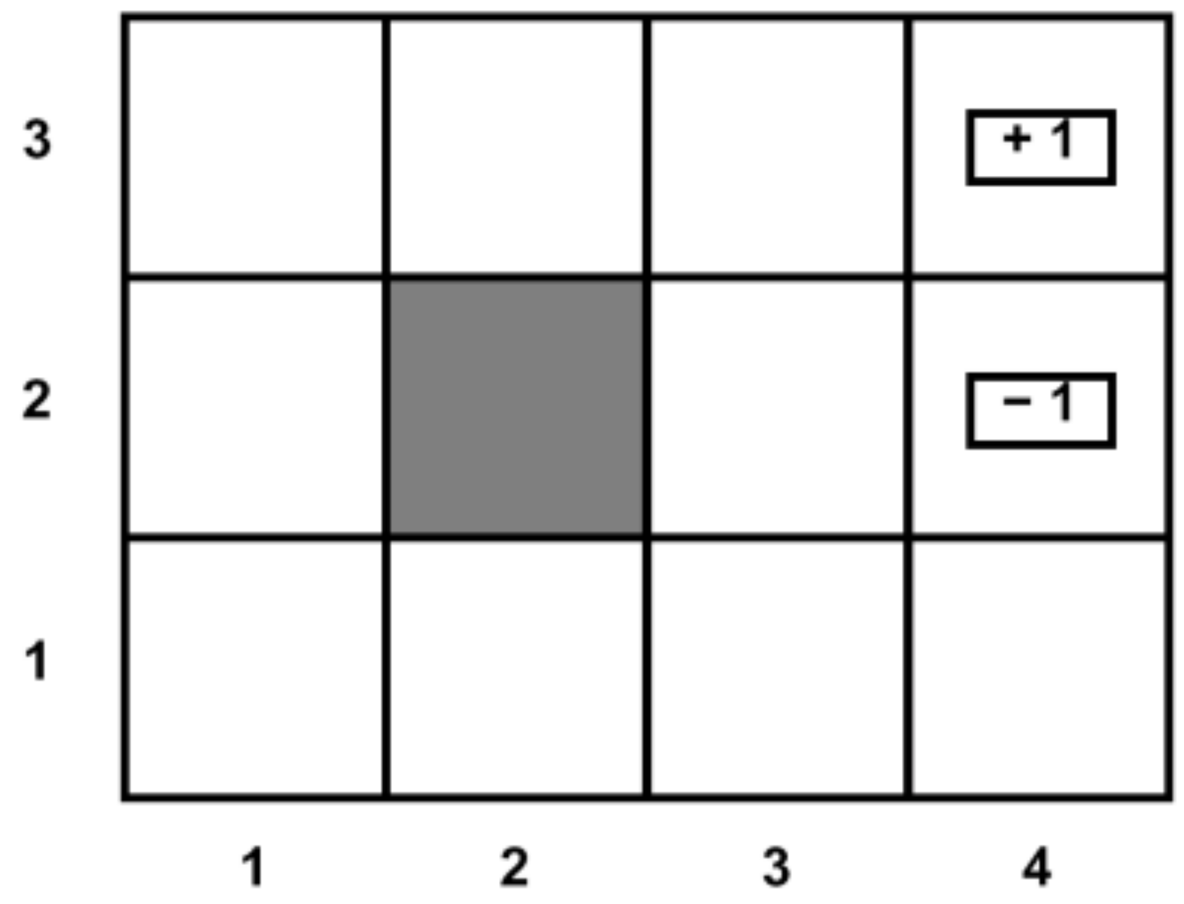

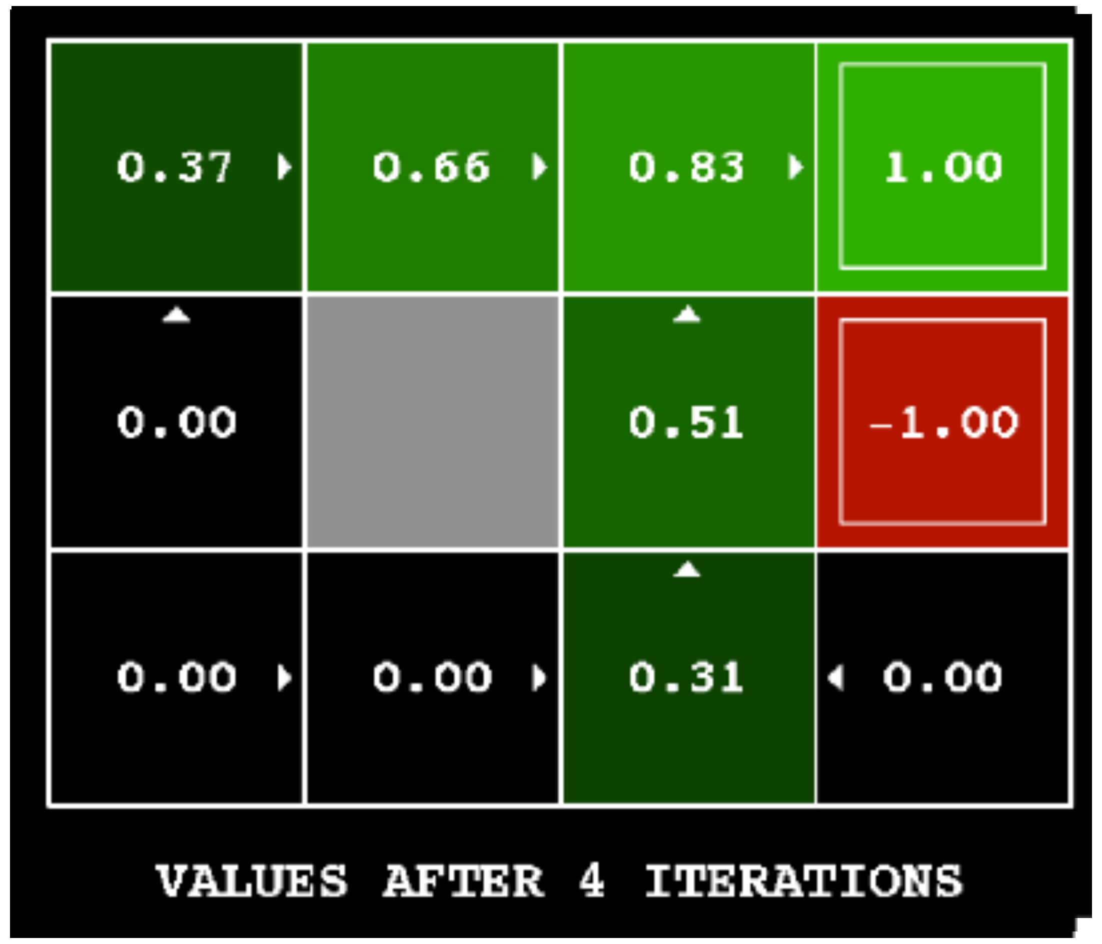

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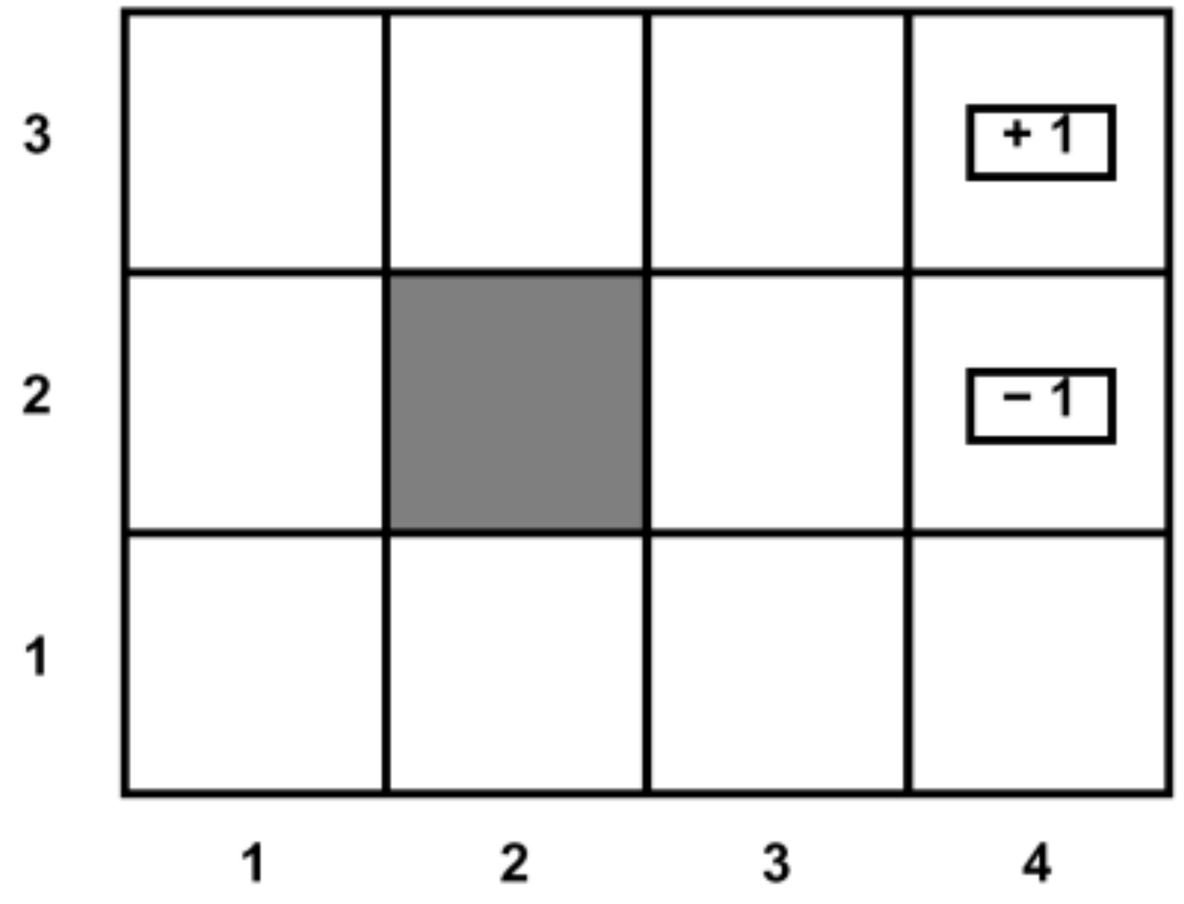

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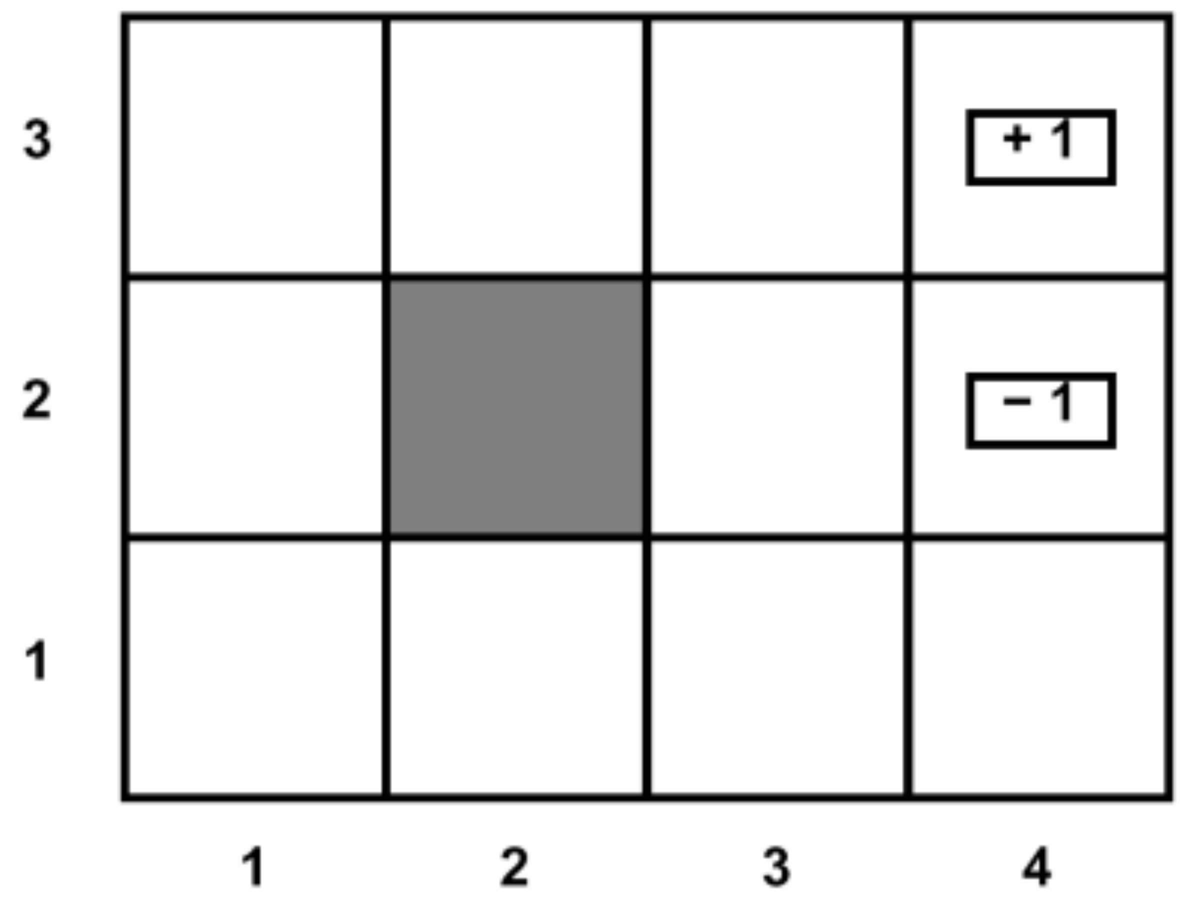

Policy iteration

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Start with π_0 (typically a random policy), and then iterate for all $s \in \mathcal{S}$ in each step

- **Policy Evaluation**

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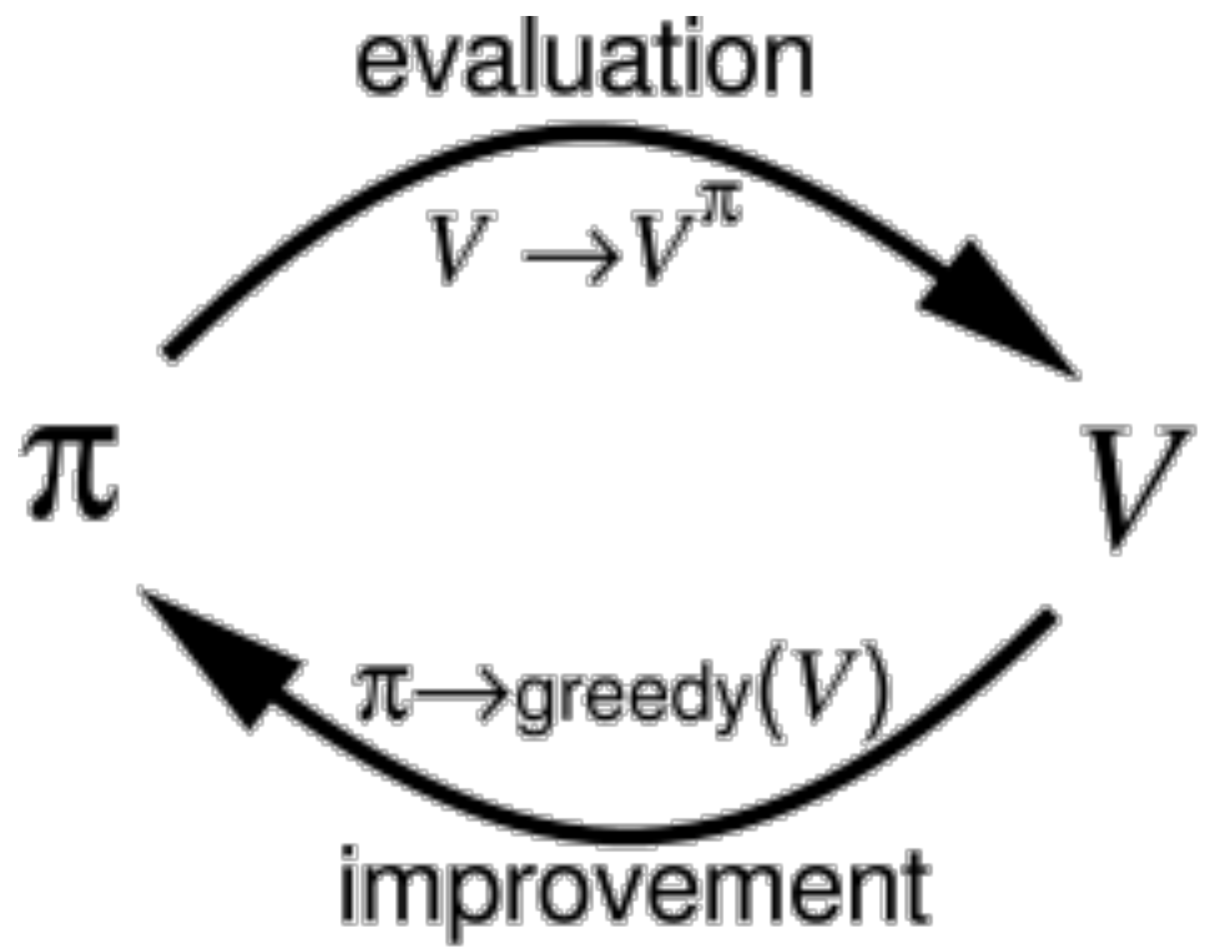

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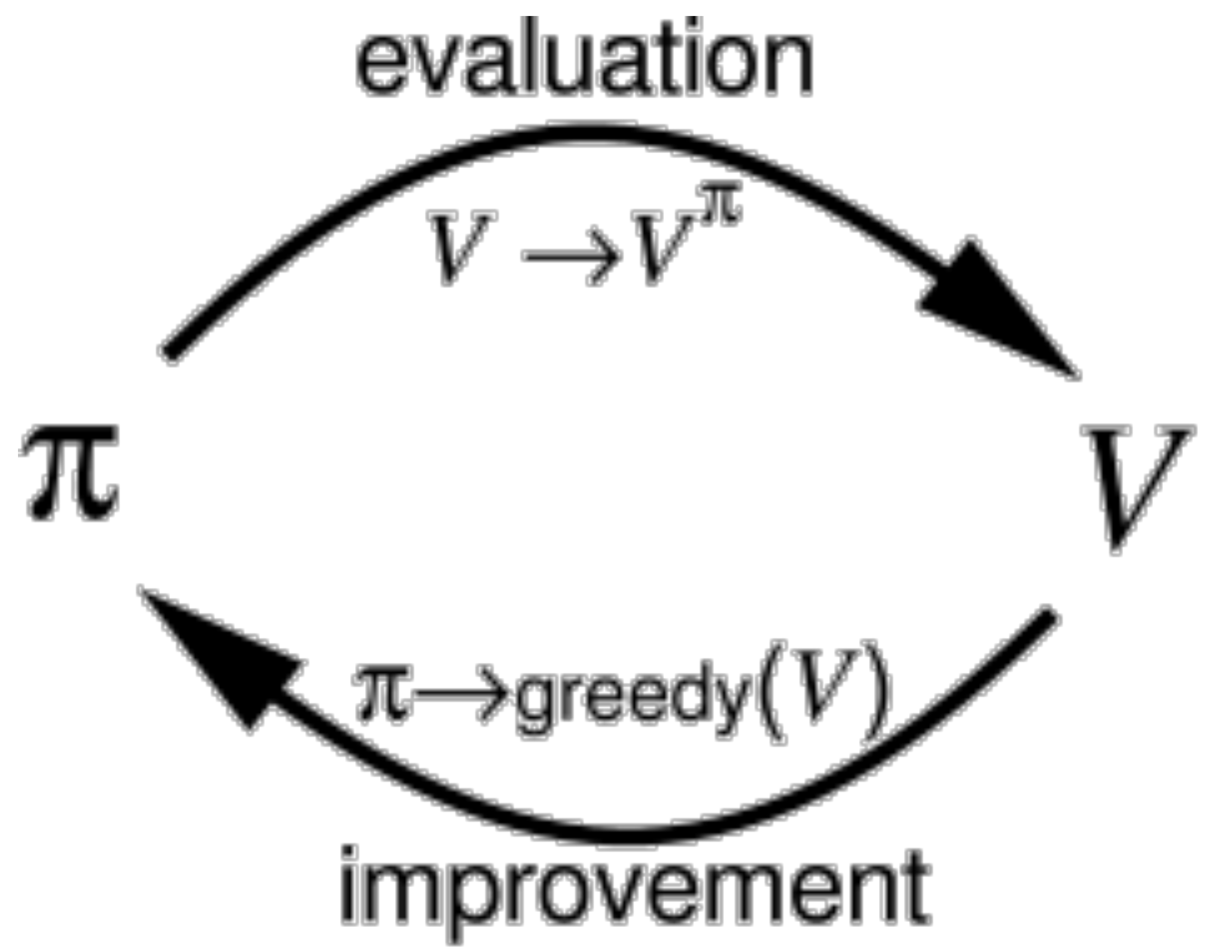

Policy iteration can converge faster than value iteration, but still requires visiting all states multiple times and lacks convergence guarantees

Challenge 2: Generalization in large action space

- What do you do when the number of states and actions are too large to visit?

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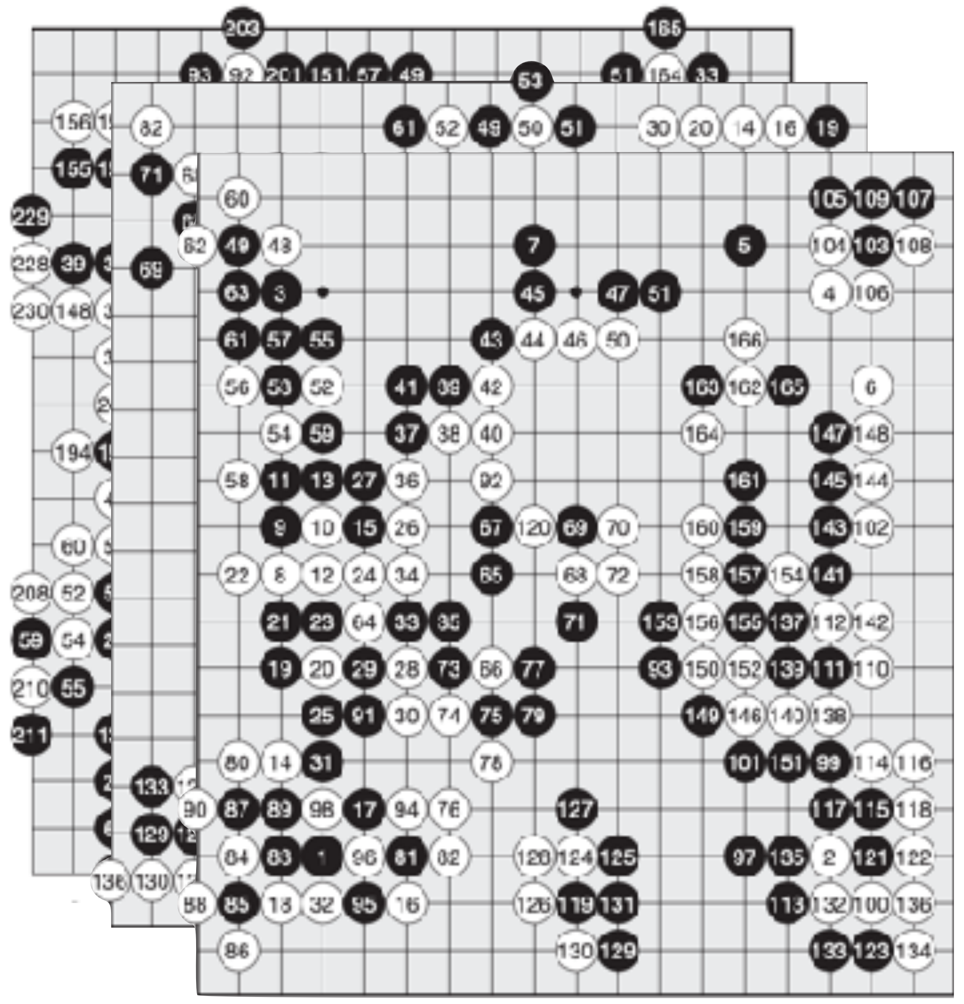
Game states

2.1×10^{170}

Atoms in observable Universe

10^{80}



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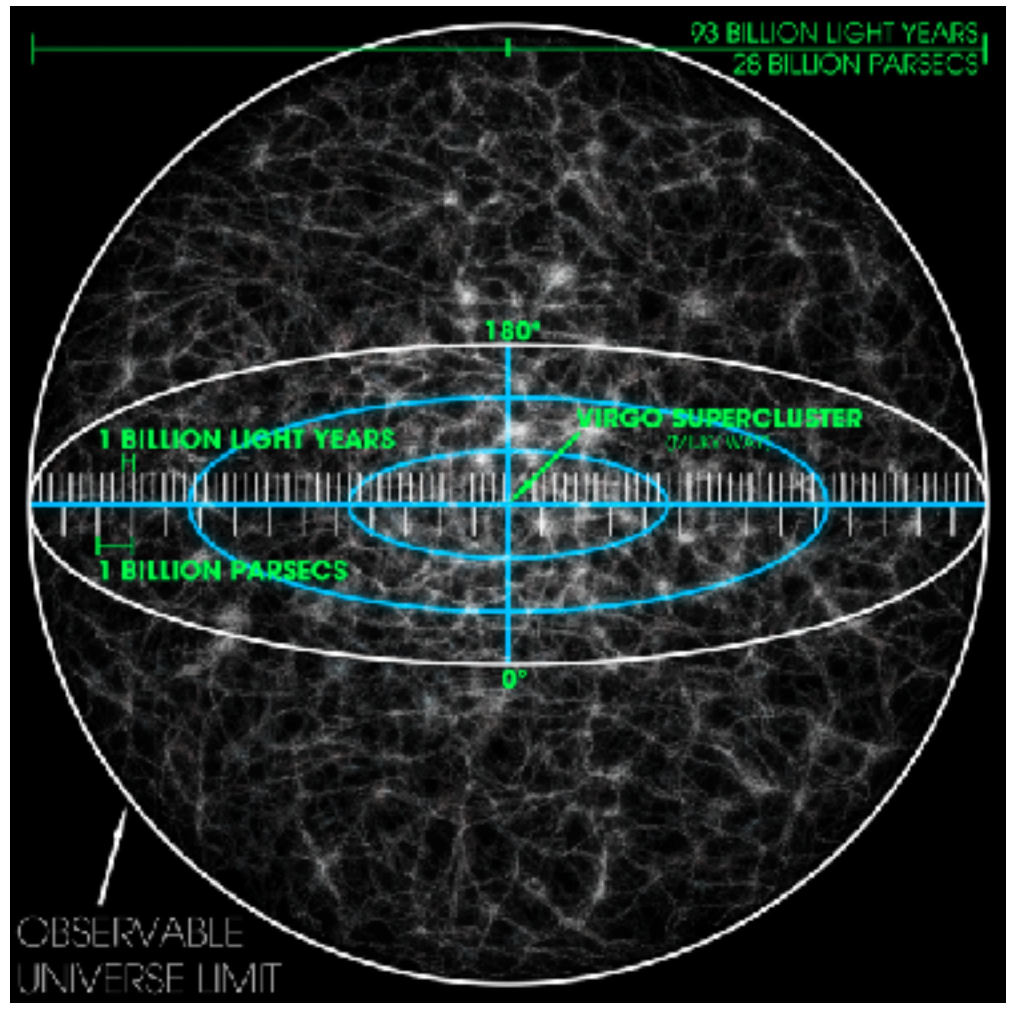

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- **Function approximation**: learn a function mapping states/actions to value, and generalize via interpolation/extrapolation

Game states

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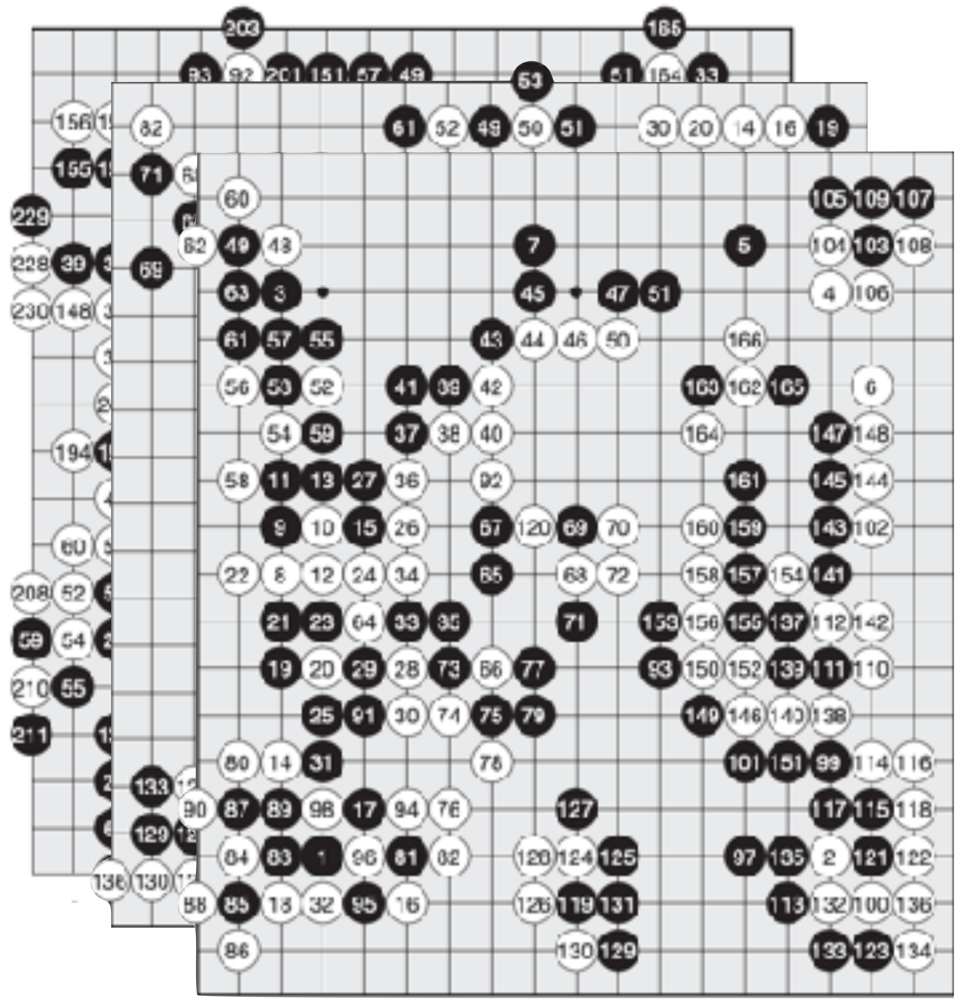

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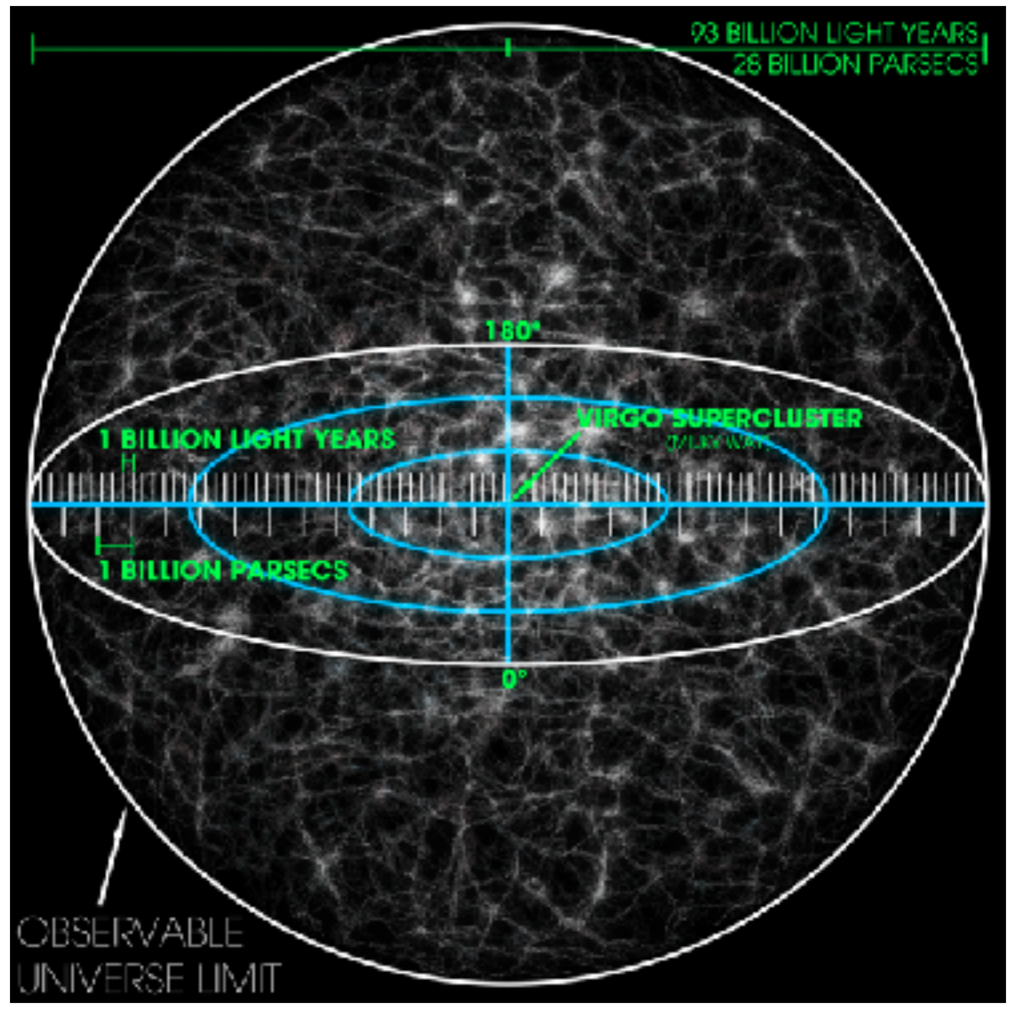
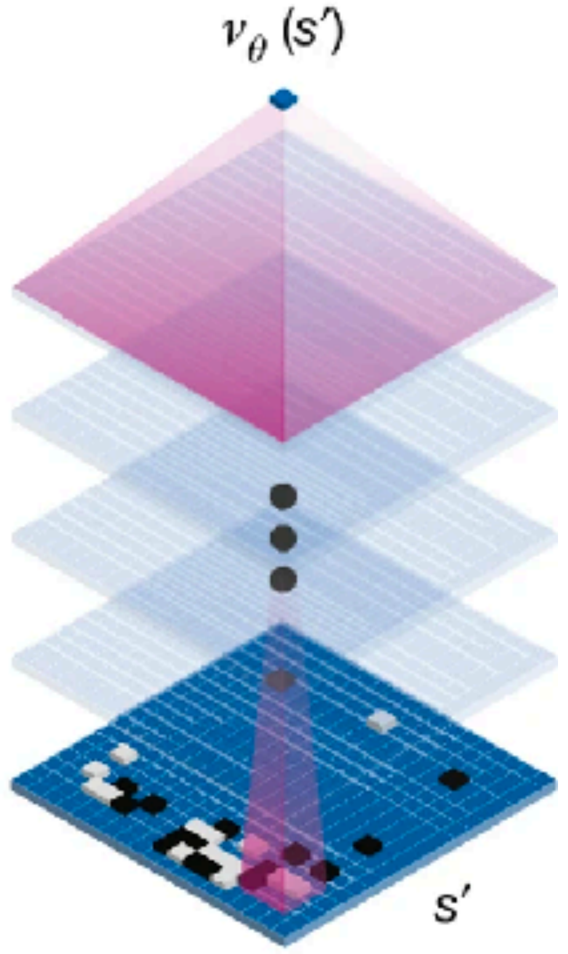
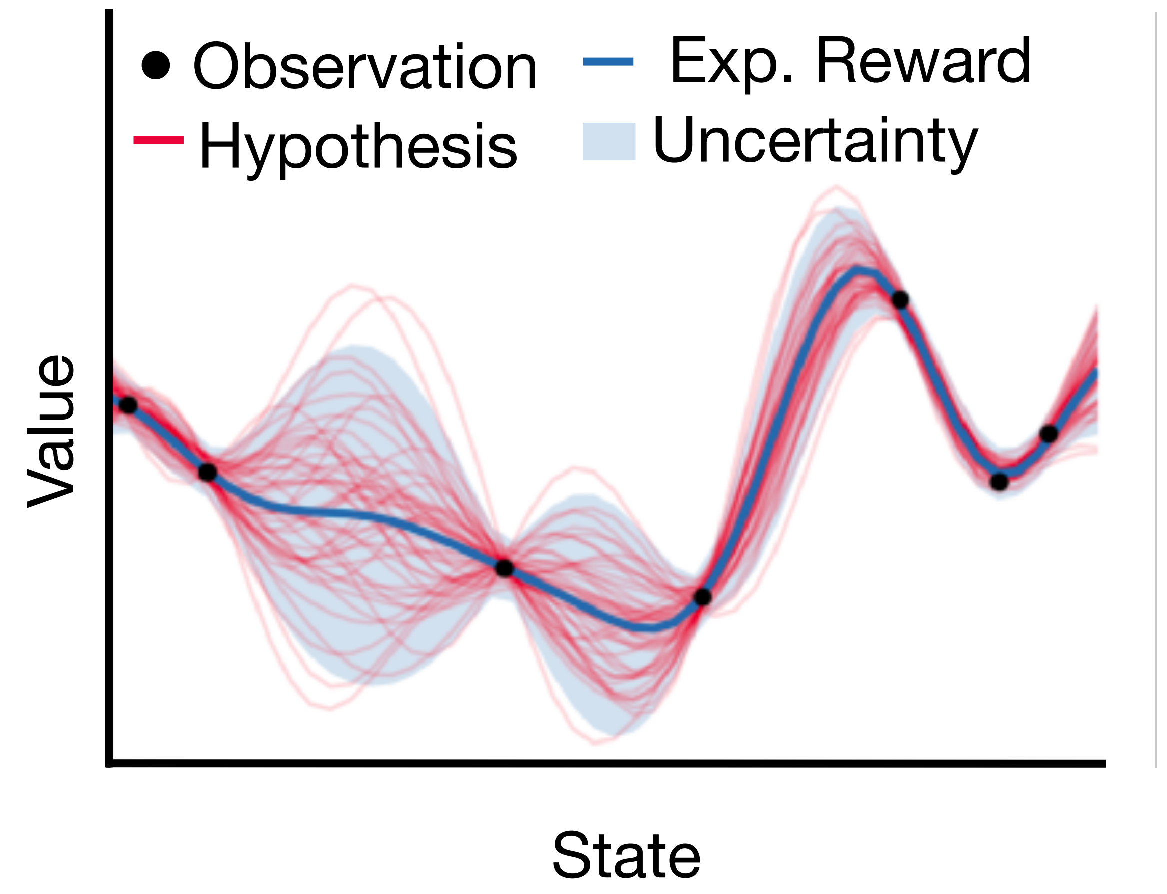


Function Approximation (Weeks 5 & 10)

$$V_\theta(s) := f(s, \theta)$$

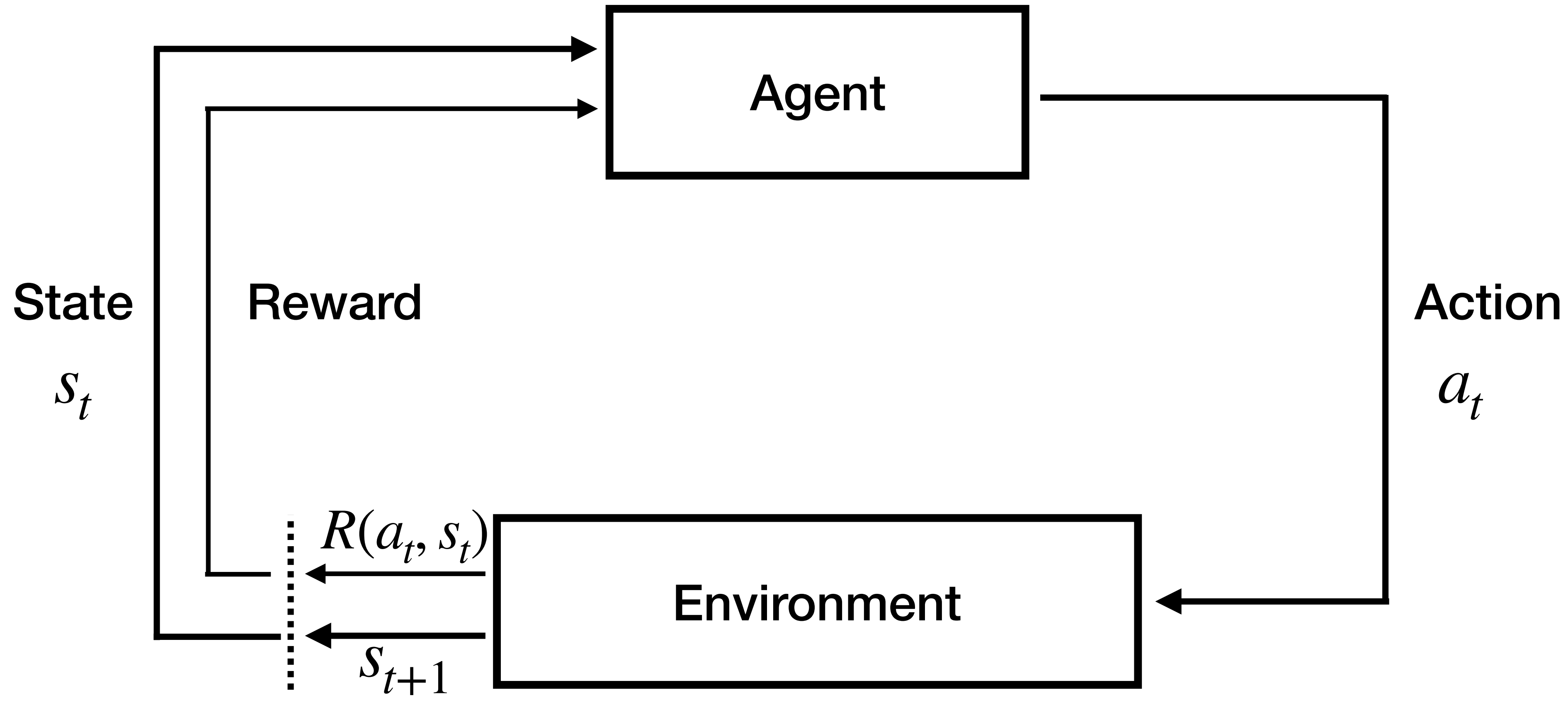


Silver et al., (*Nature* 2016)



Wu et al., (*AnnRevPsych* 2024)

RL summary



- **Normative** framework for learning an optimal policy π^* in arbitrarily complex environments
    - Simplest setting is a 2-armed bandit problem, where Q-learning is equivalent to Rescorla-Wagner
    - More complex settings require *credit assignment* and *generalization*
- RL also provides a **descriptive** model of human learning
    - TD-learning prediction error tracks dopamine signals in the brain and widely used to study human behavior (more on this next week)

5 minute break

Model-free RL ⟷ Model-based RL

Model-free RL

- Habit
- Cheap
- $Q(s, a)$
- Myopically selecting actions that have been associated with reward

Model-based RL

- Goal-directed
- Computationally costly
- $P(s', r \mid s, a)$
- Planning and seeking of long term outcomes

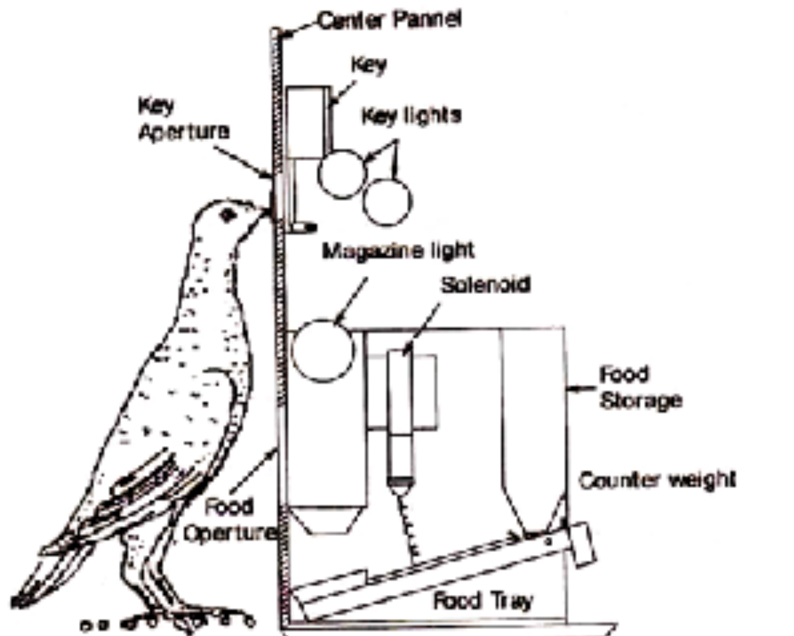
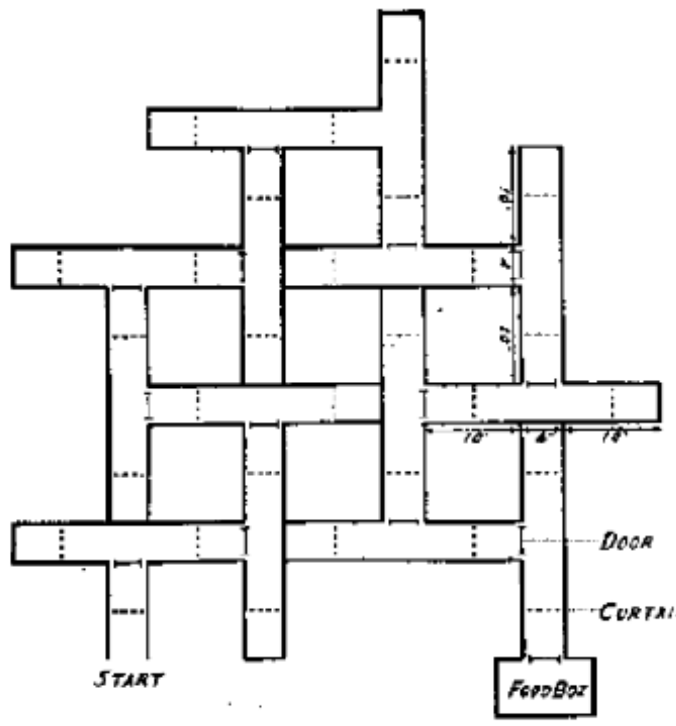
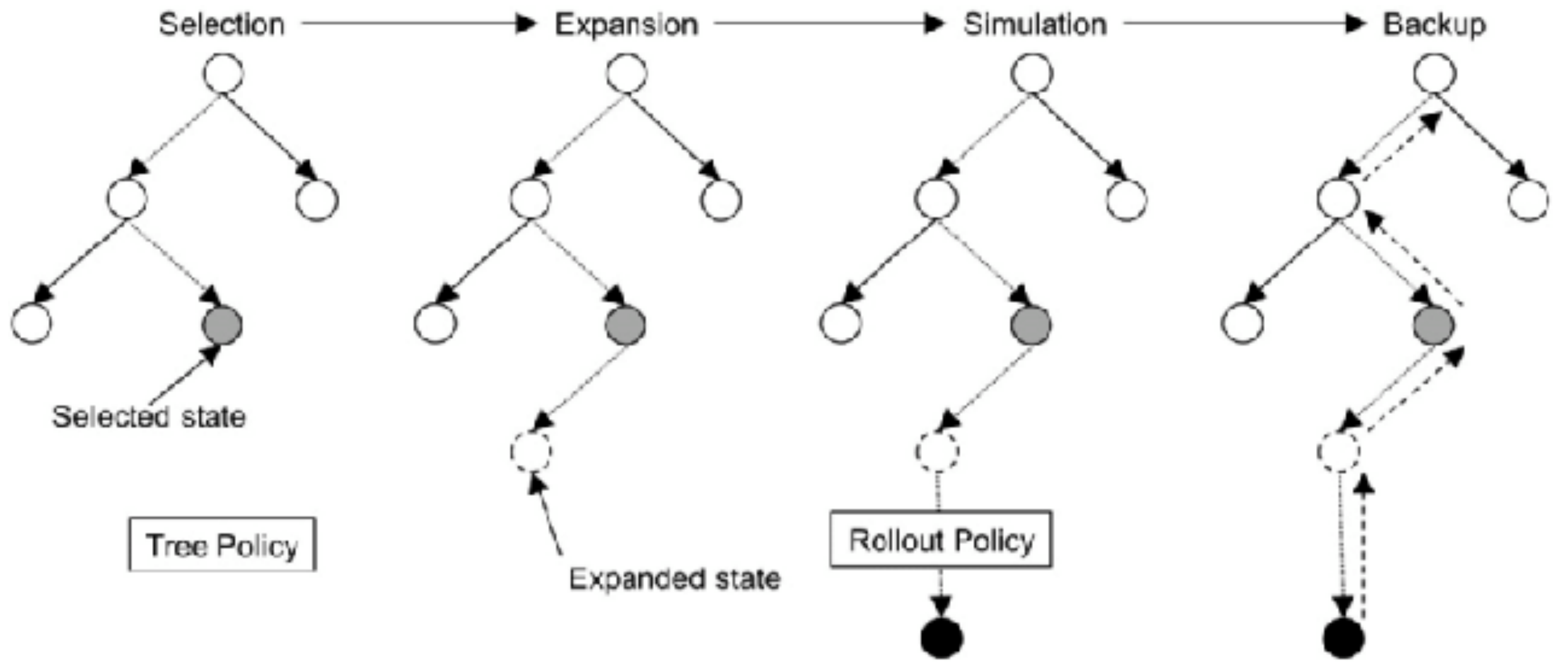


Illustration. Skinner box as adapted for the pigeon.



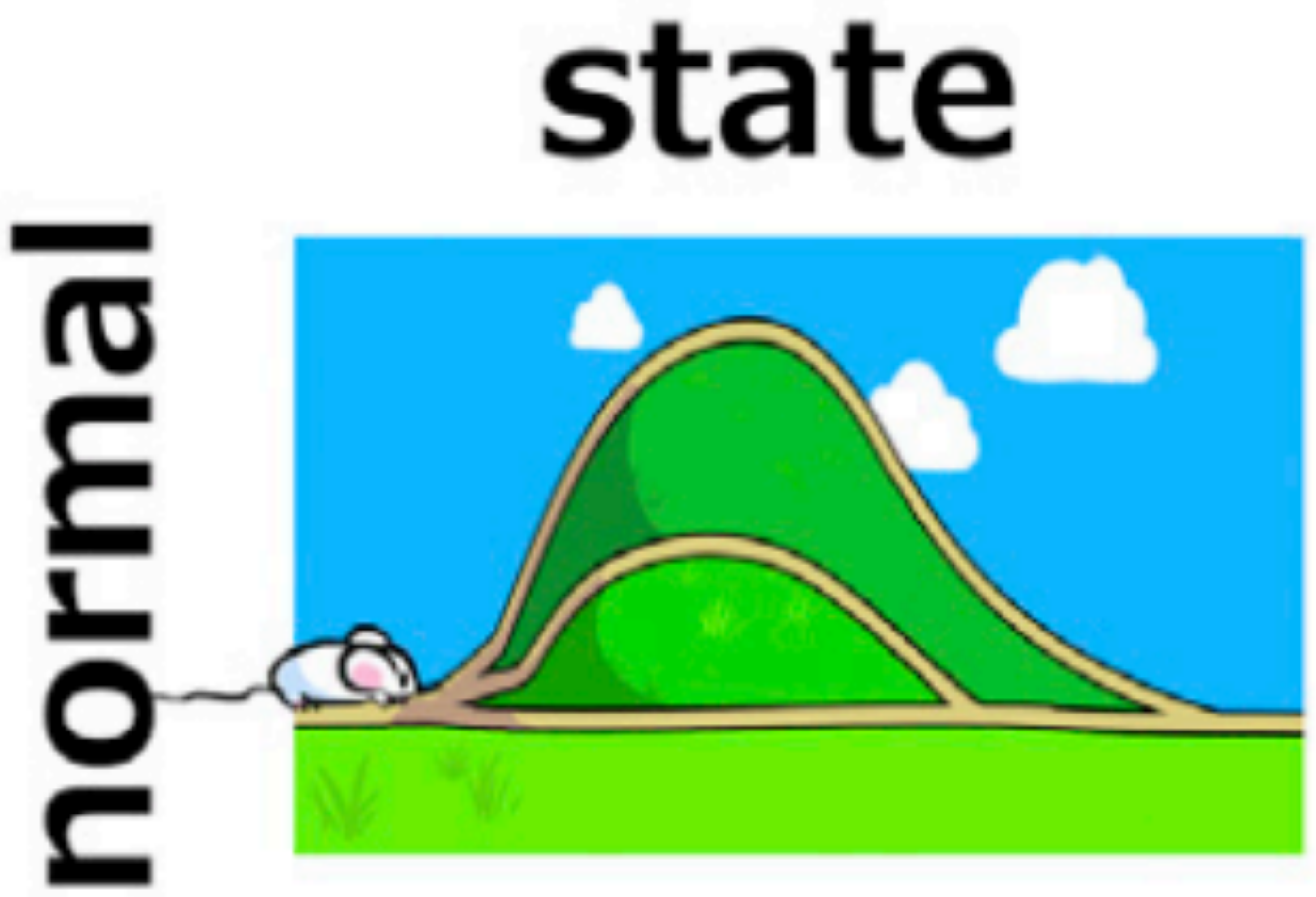
Plan of maze
14 Unit T-Alley Maze
FIG. 1
(From M. H. Elliott, The effect of change of reward on the maze performance of rats. Univ. Calif. Publ. Psychol., 1928, 4, p. 22.)

Monte carlo tree search



Duarte et al,. (2020)

image credit Alyssa Dayan (from Dolan & Dayan, 2013)



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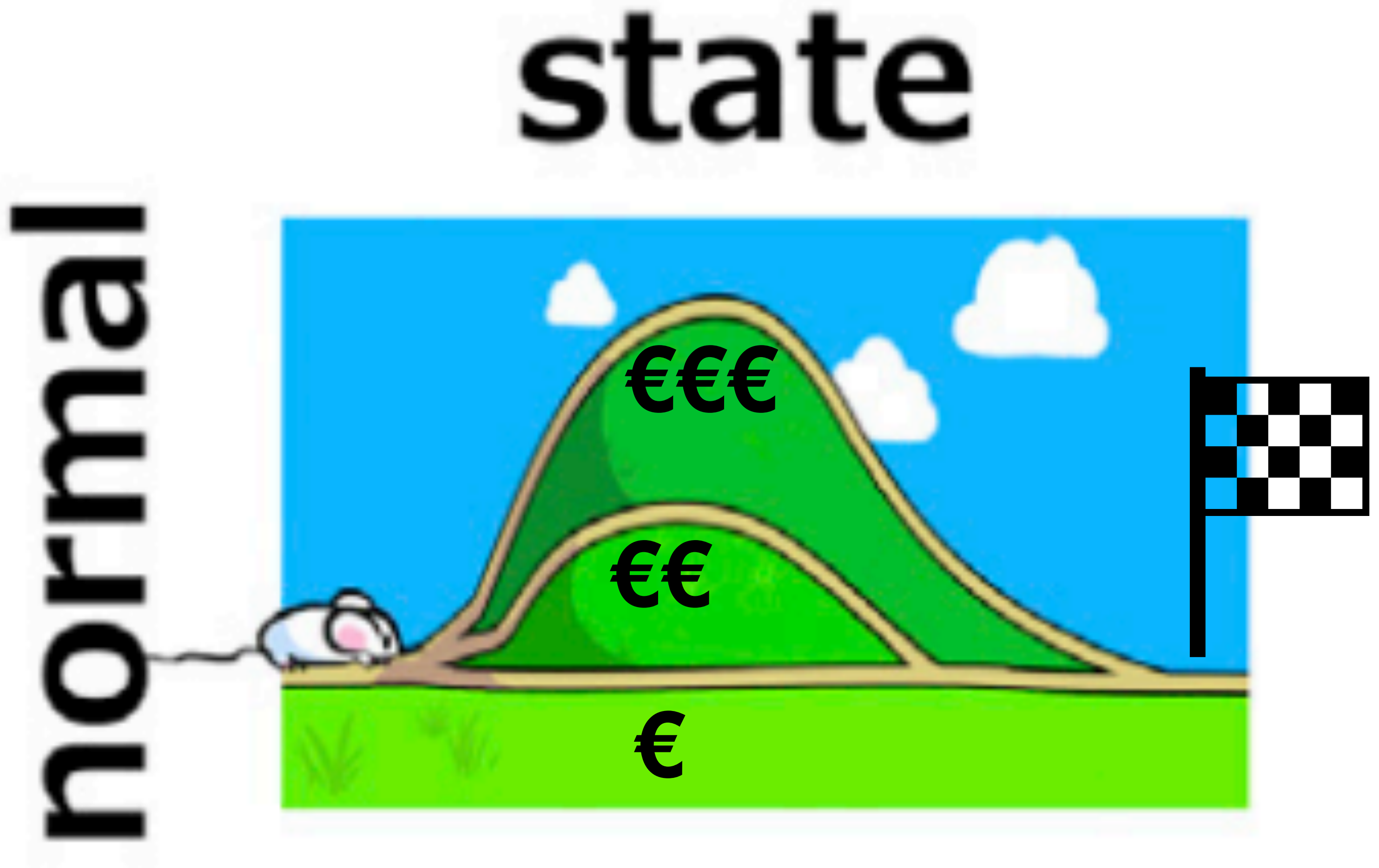

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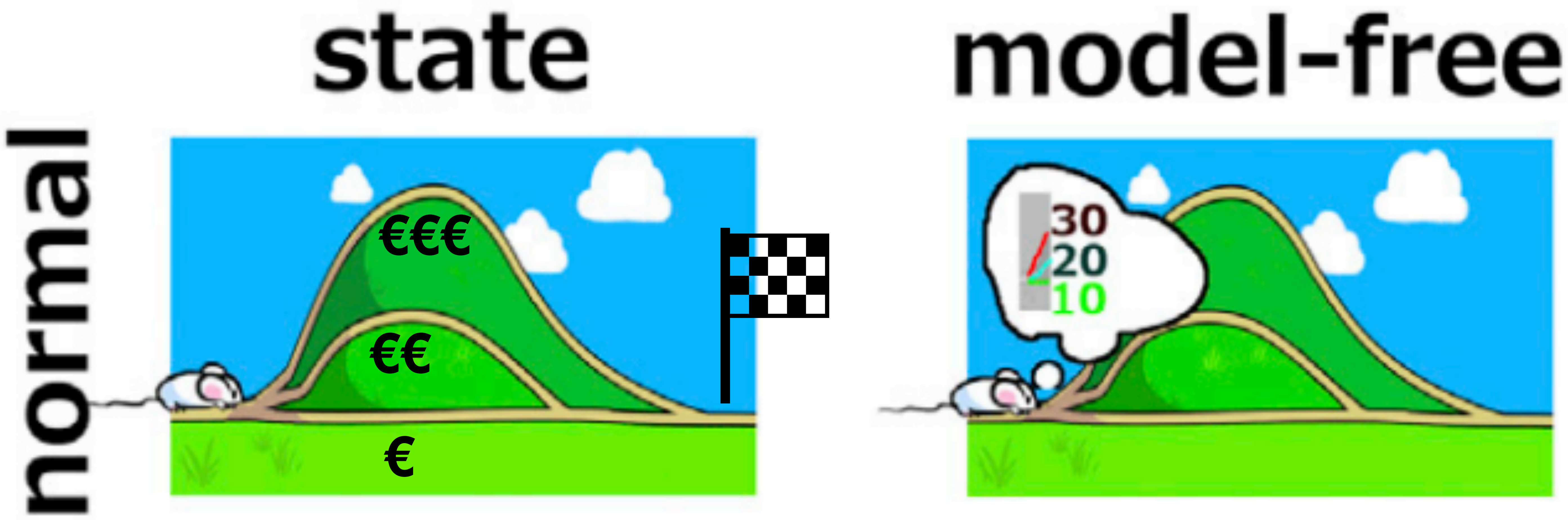

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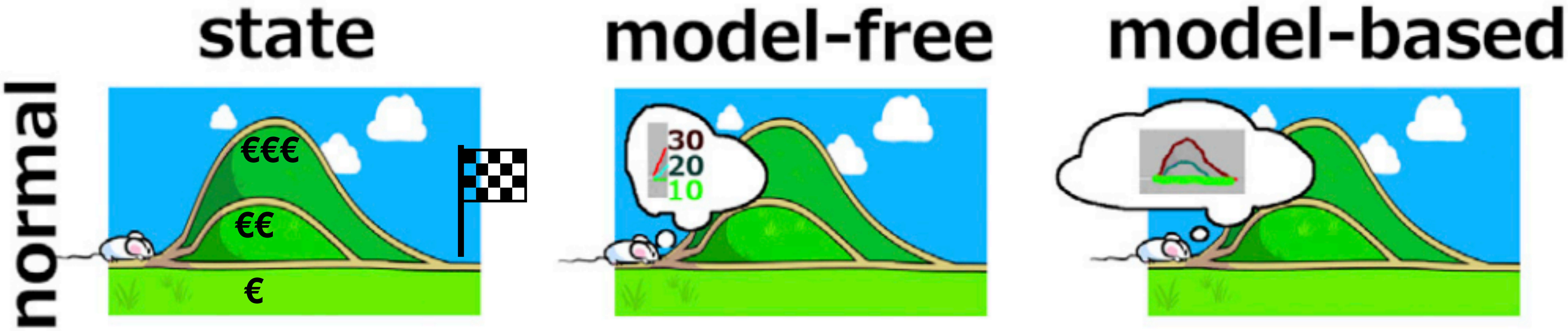

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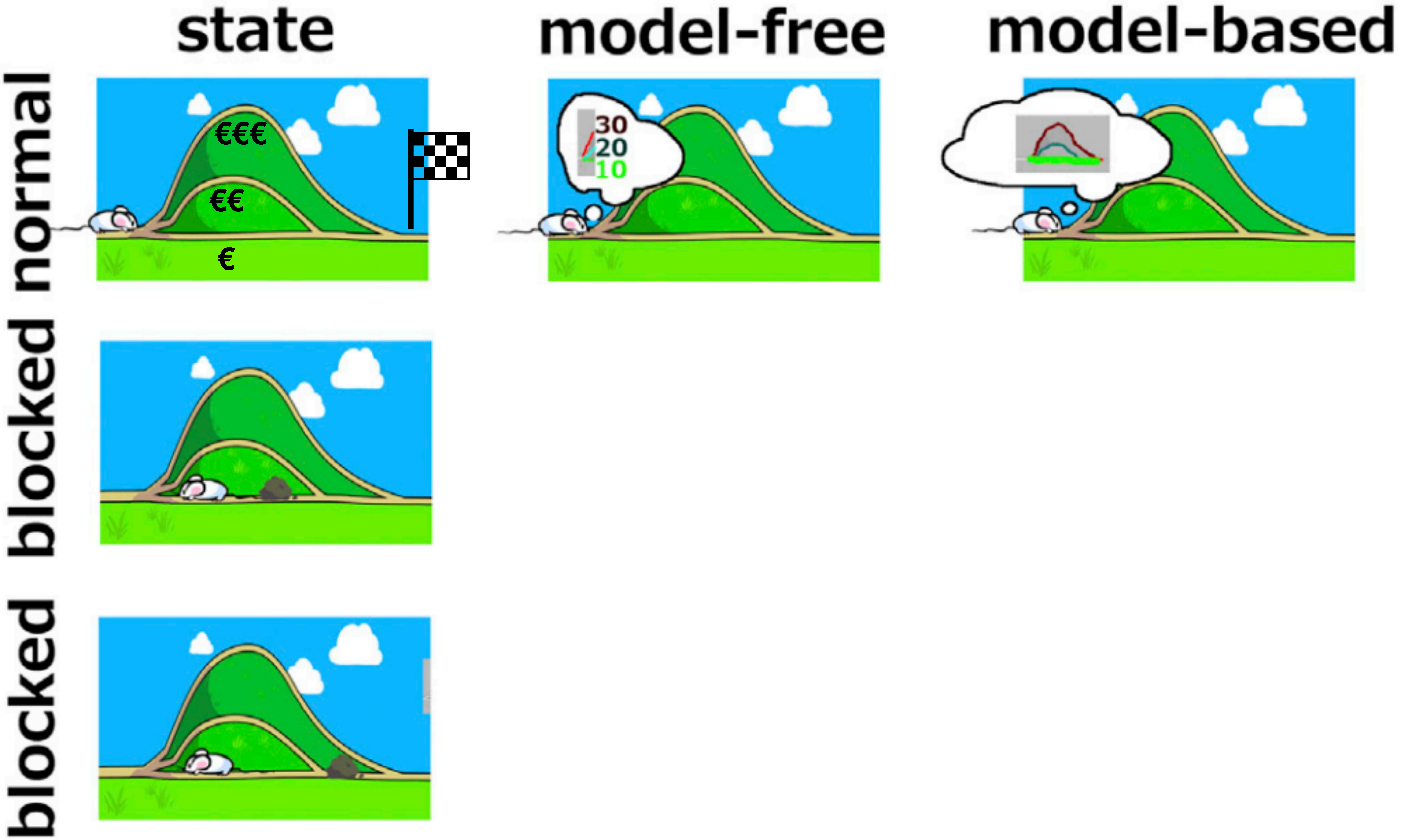

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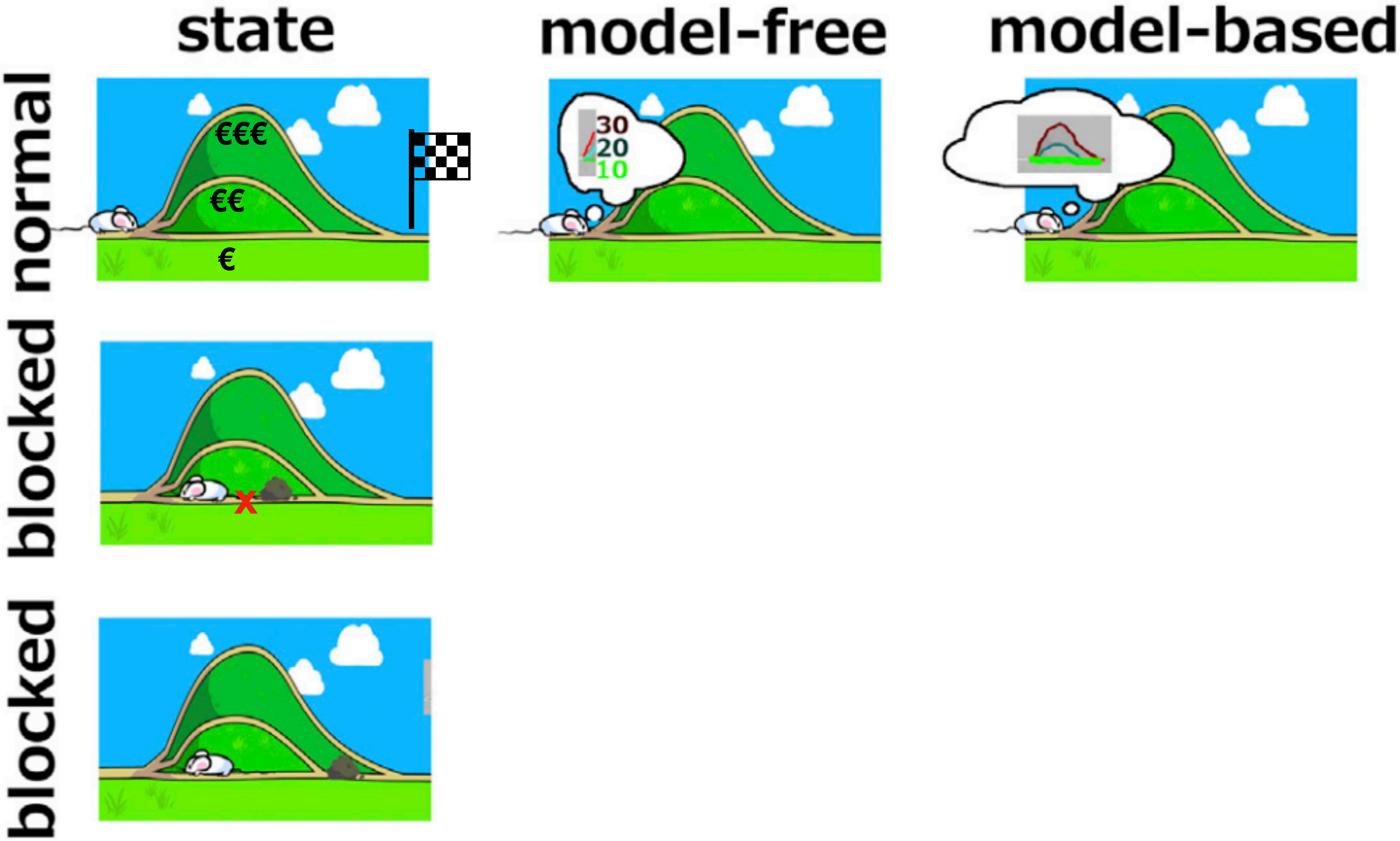

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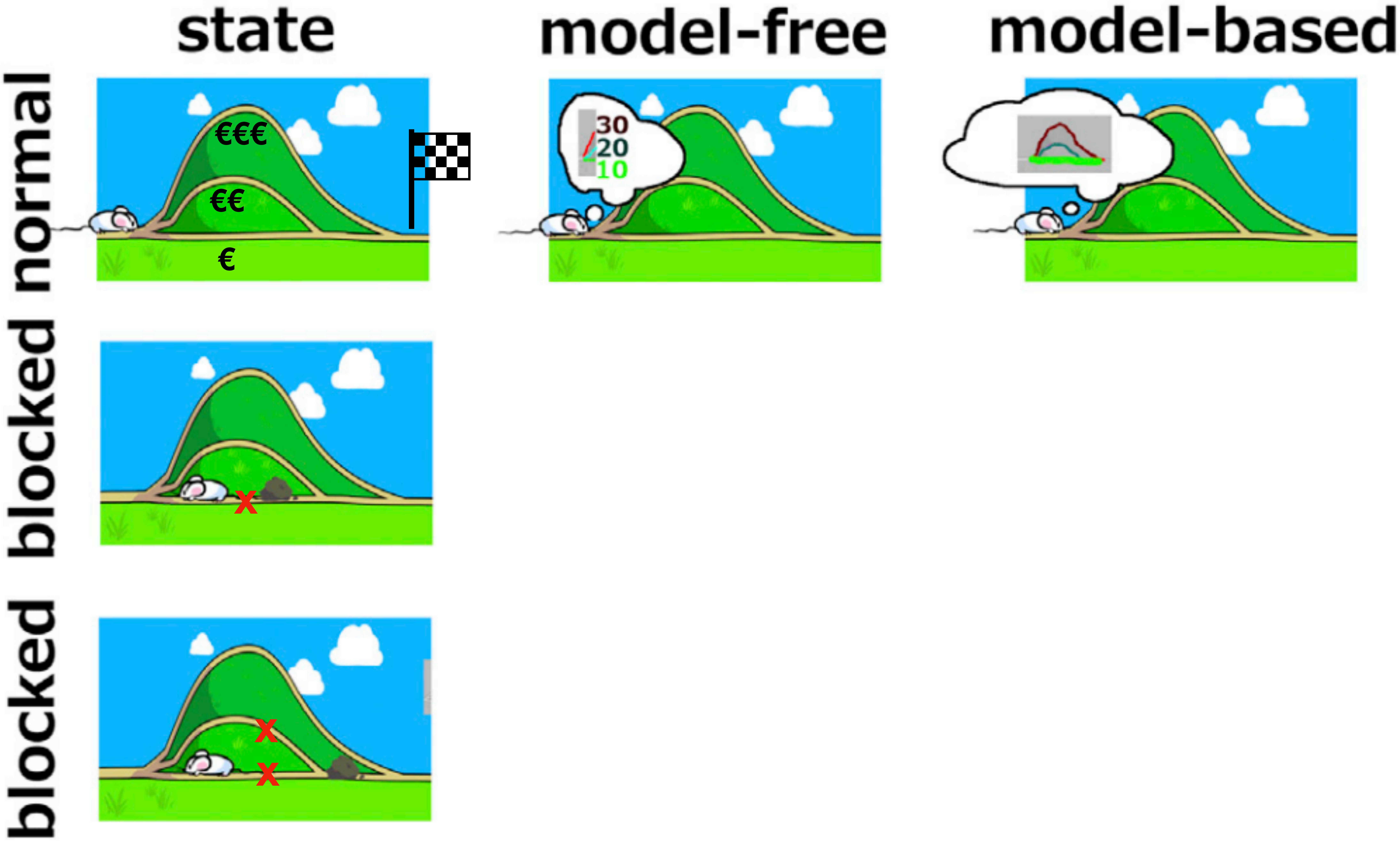

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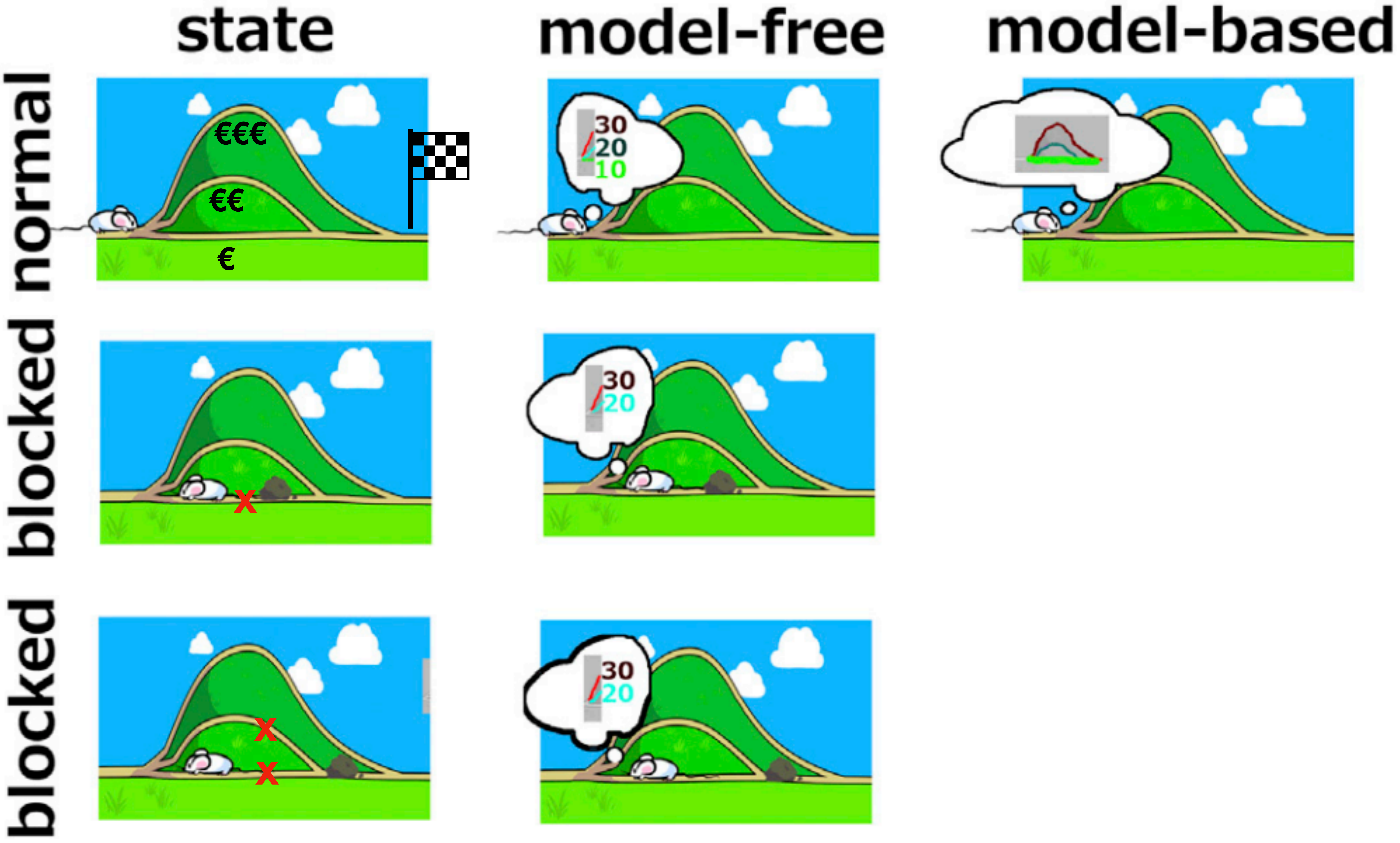

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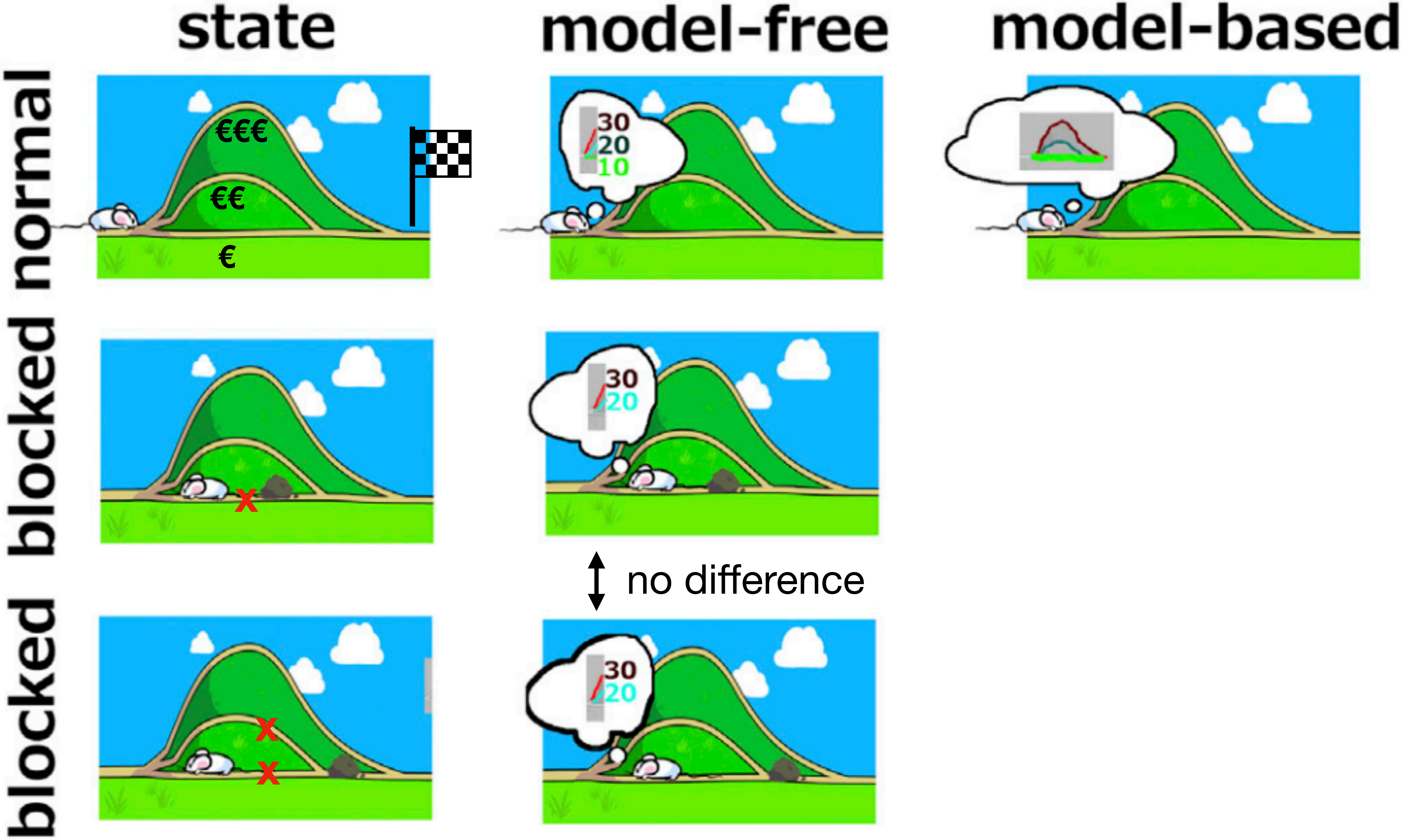

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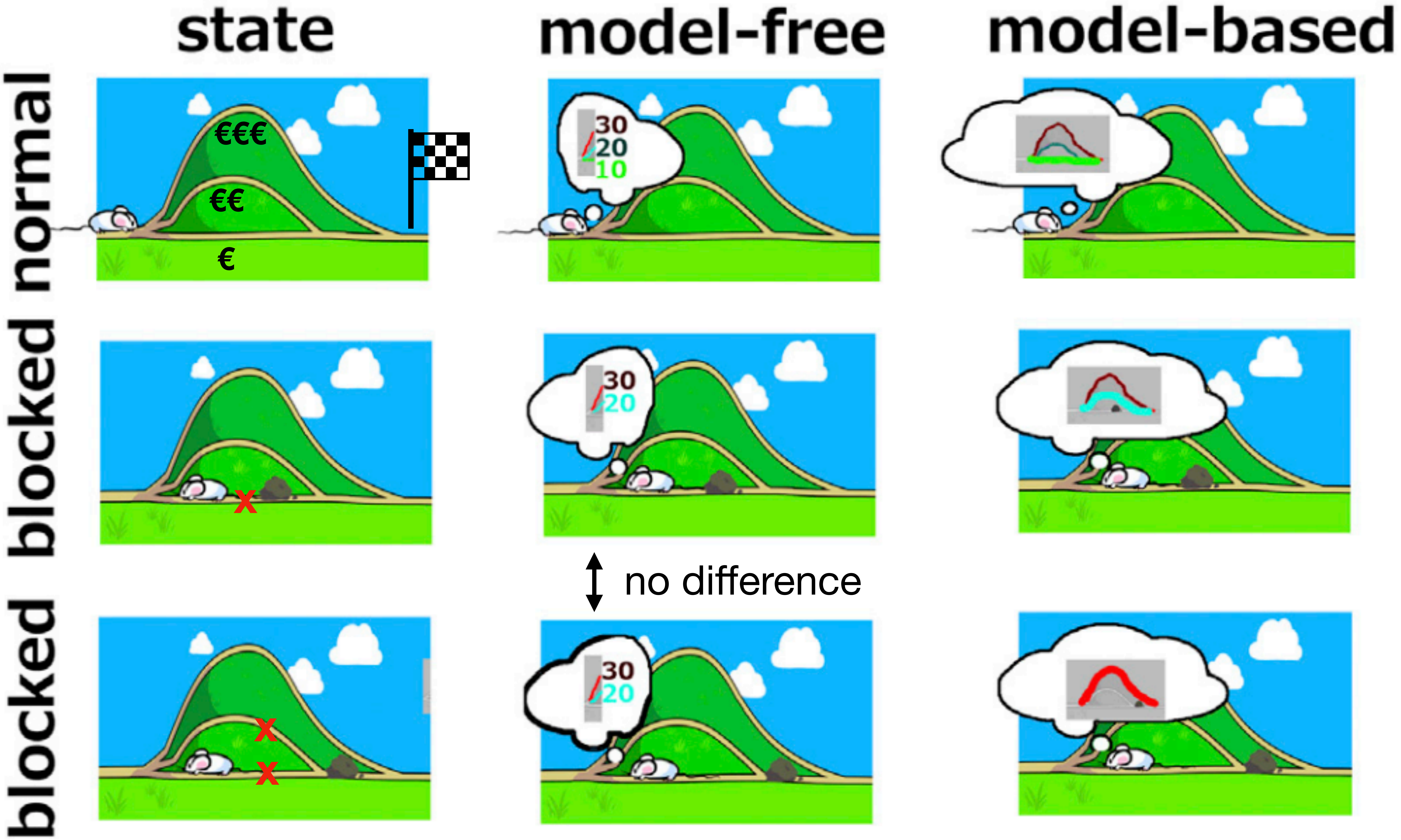

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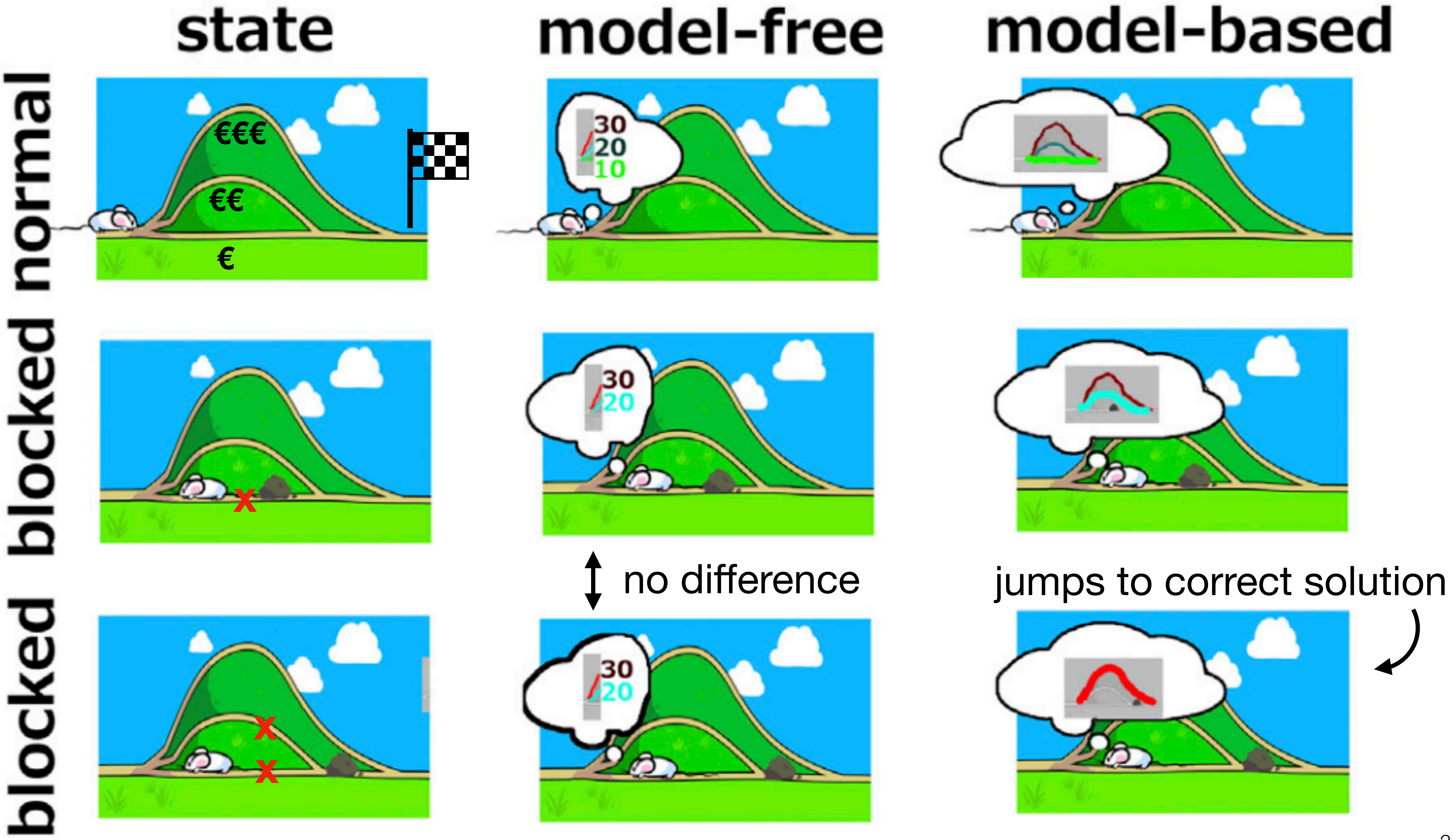
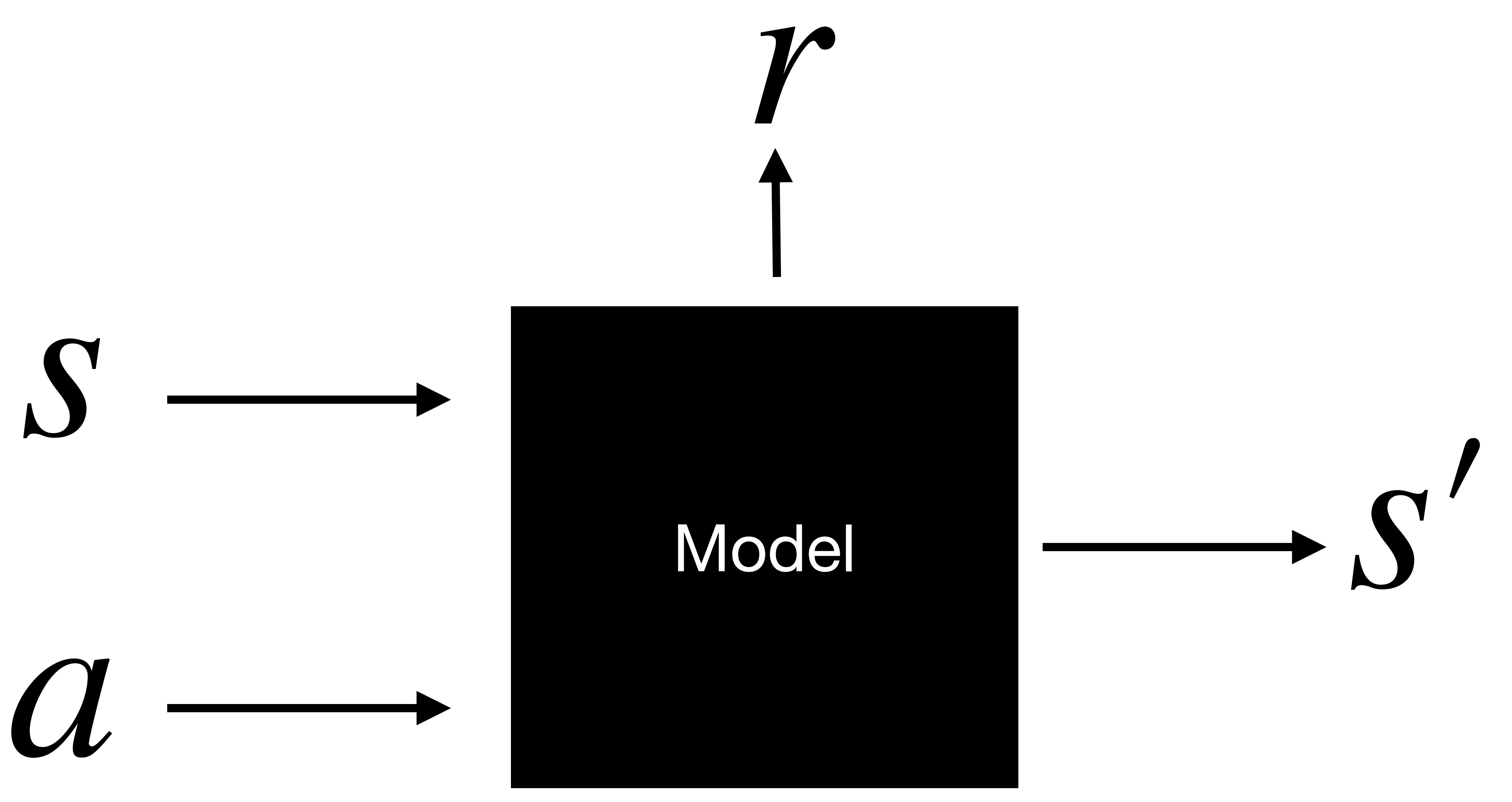
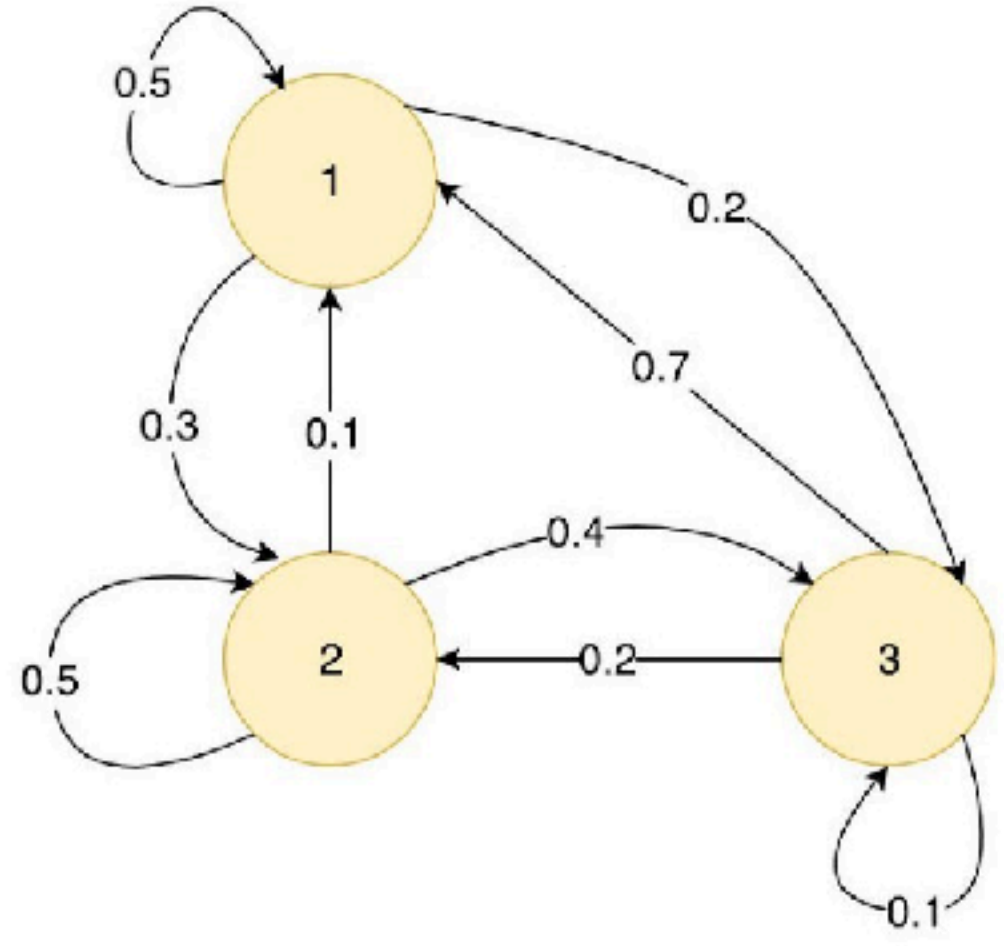


What is model-based RL?

- An internal representation of the environment
- Ingredients:
  - Transition matrix $T(s' \mid s, a)$
  - Reward function $R(s, a)$
  - State space $s \in \mathcal{S}$
  - Action space $a \in \mathcal{A}$
- How is it learned? (find out next week!)



MDP



Transition Matrix

$$\begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \end{array} \begin{array}{c} \begin{array}{ccc} s_1 & s_2 & s_3 \end{array} \\ \begin{bmatrix} 0.5 & 0.1 & 0.7 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.4 & 0.1 \end{bmatrix} \end{array}$$

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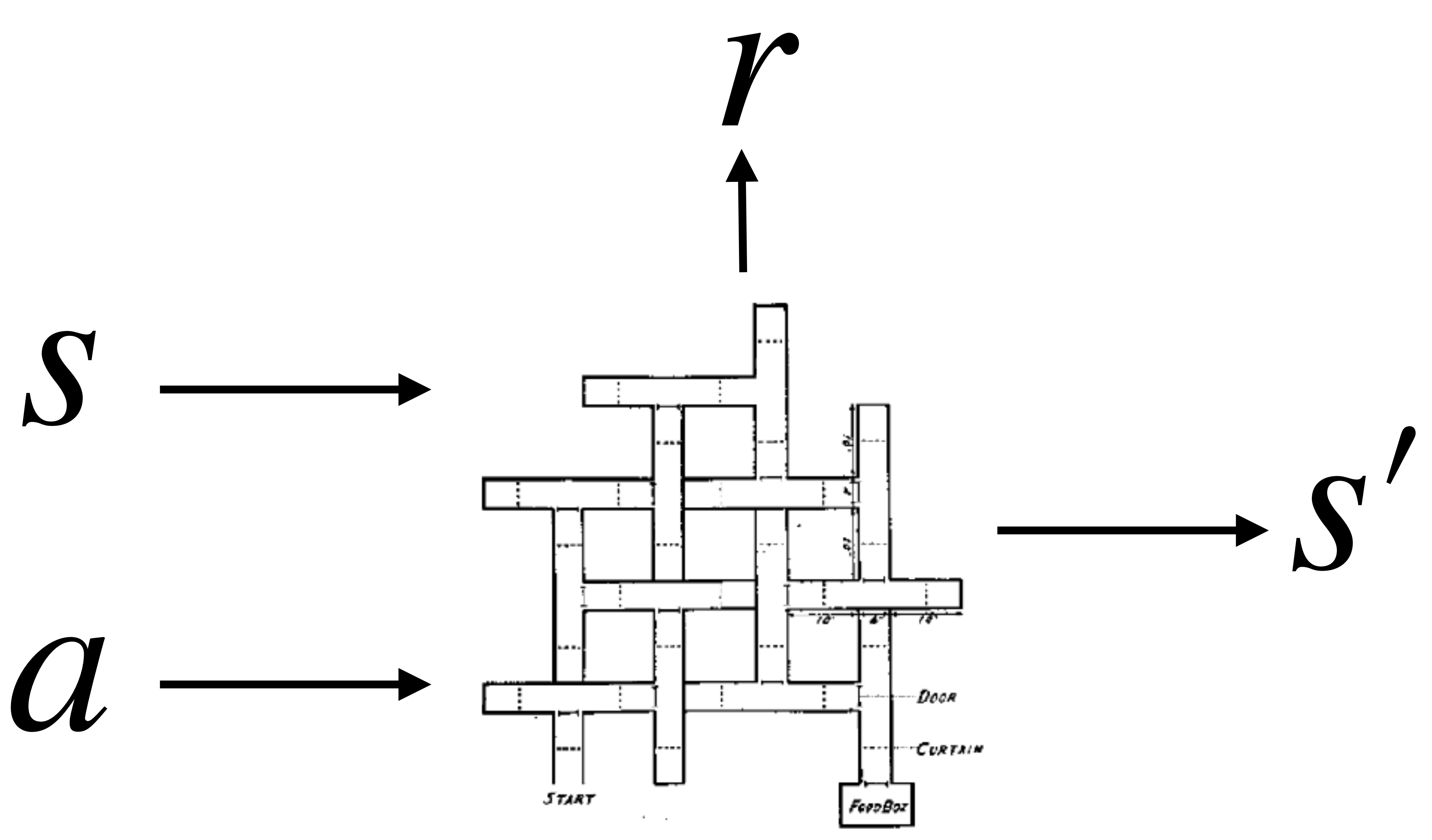

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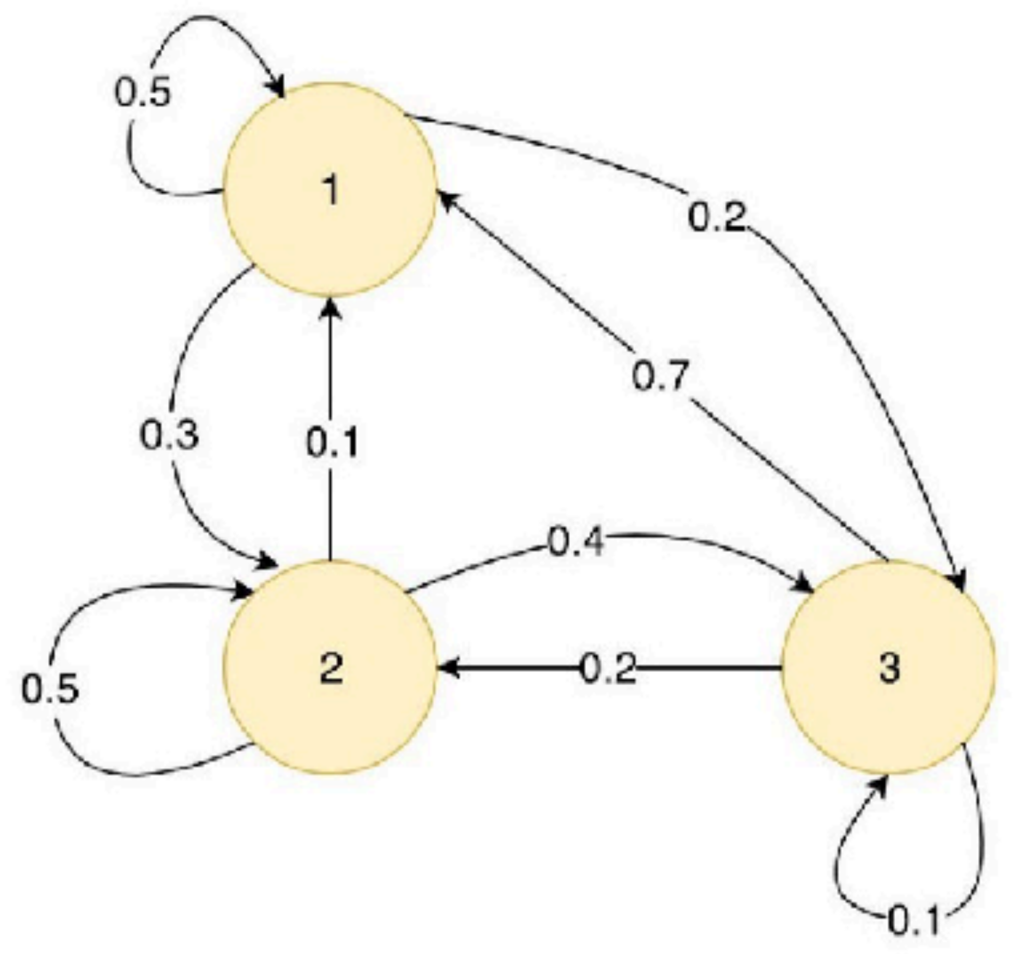

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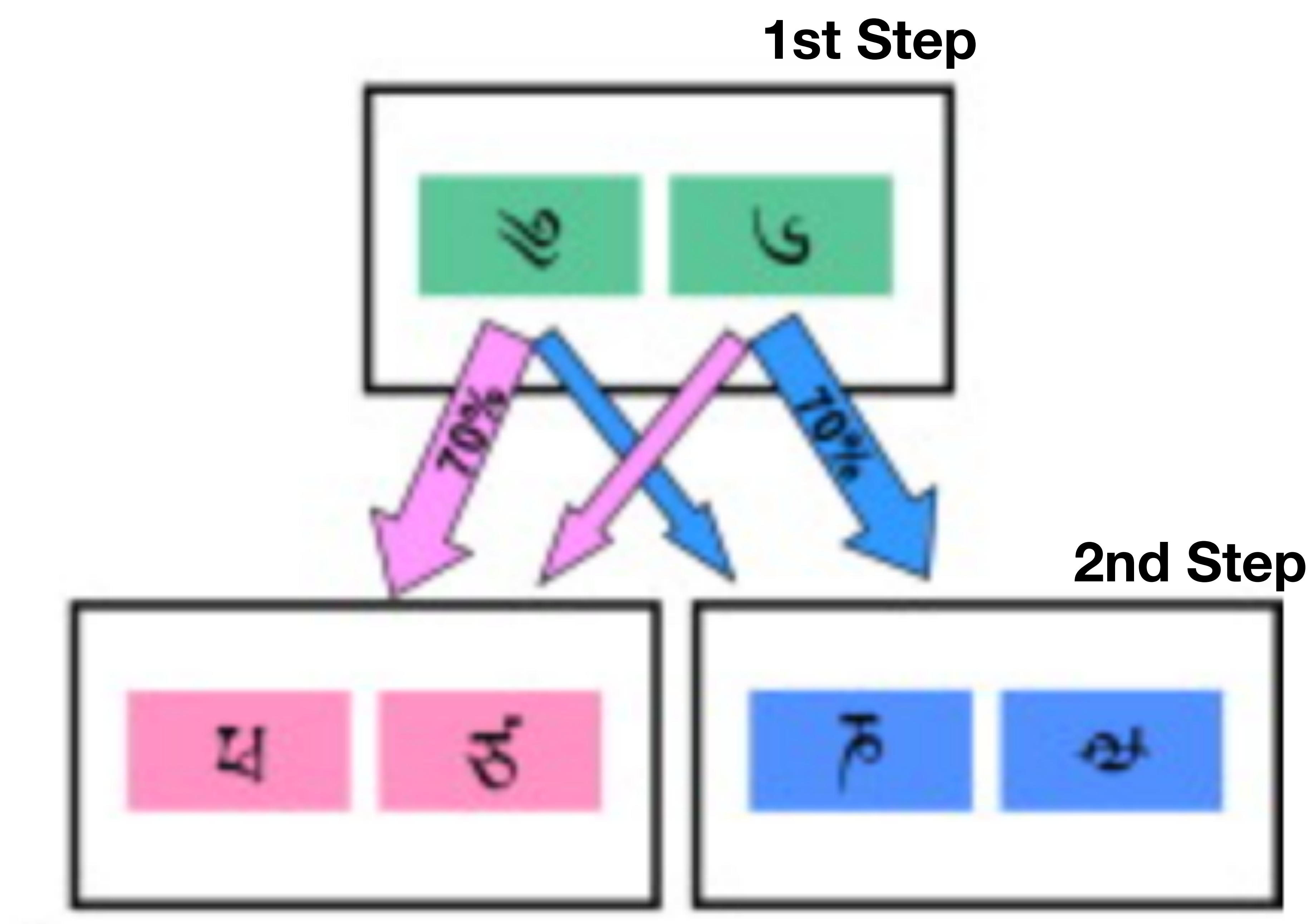
Daw et al., (2011)

Two-step task

- Two-stage decision-making task used to distinguish model-free vs. model-based learning

- 1st step choices have common (70%) and rare (30%) transitions to different sets of 2nd step options

- 2nd step options have different P(reward)



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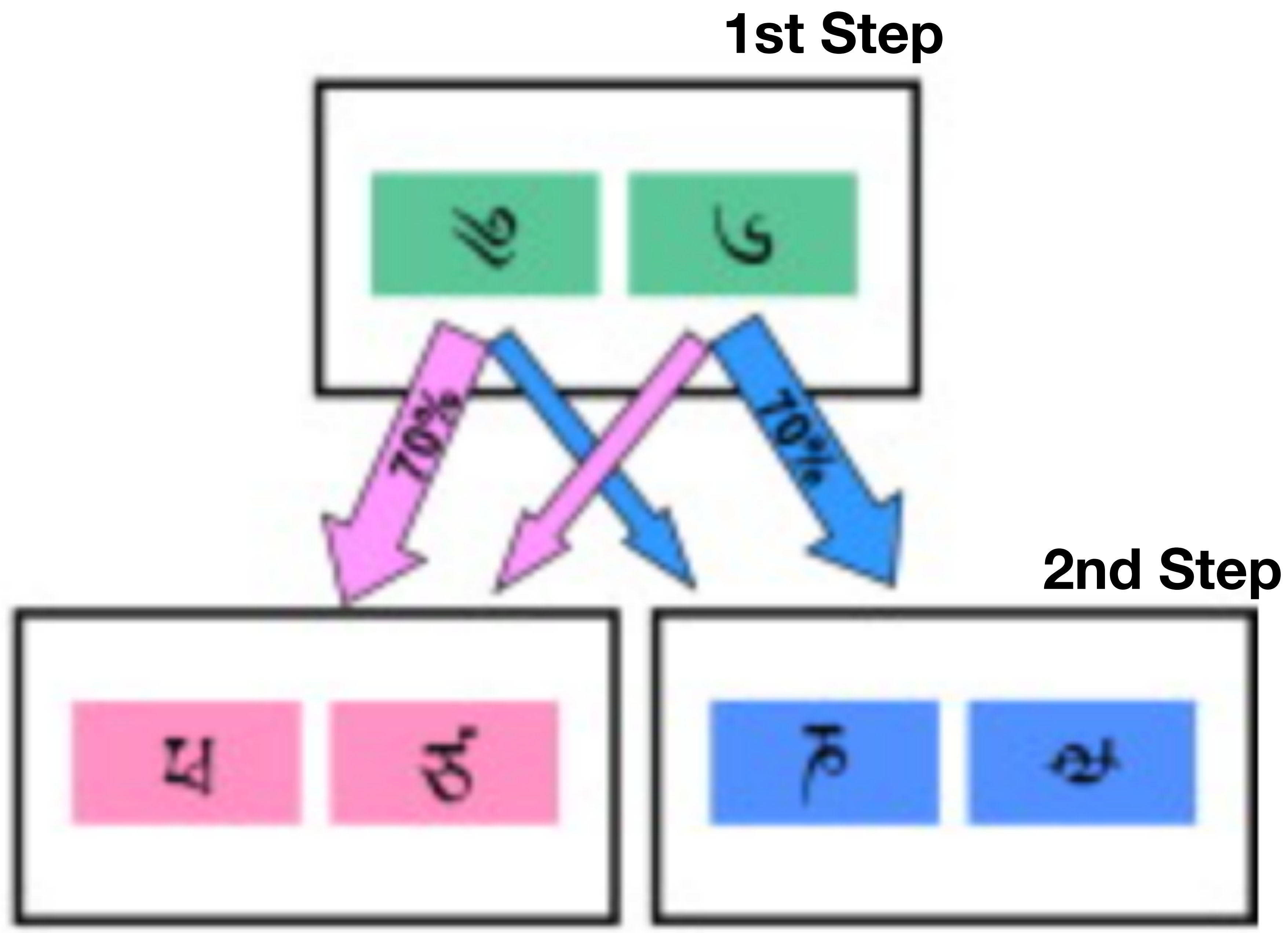

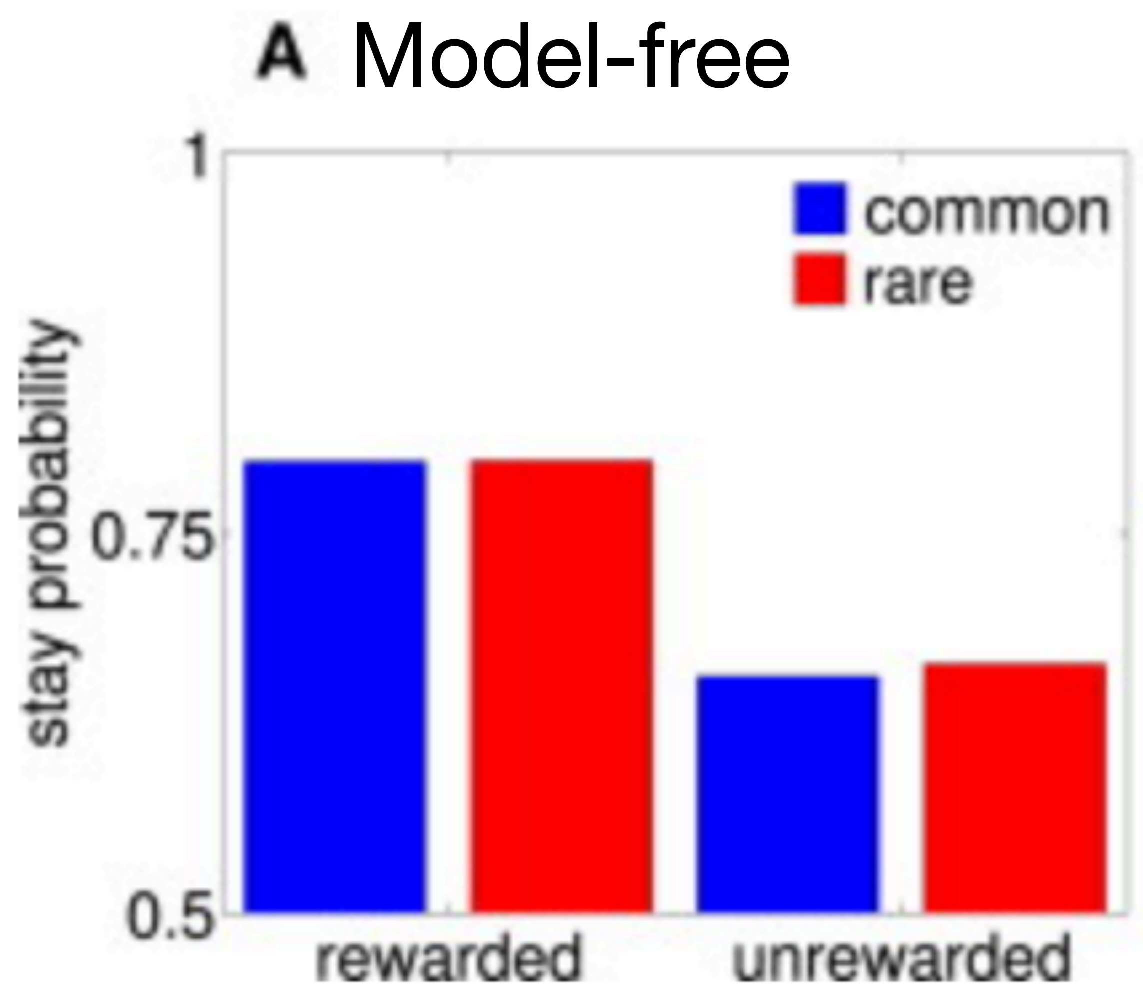

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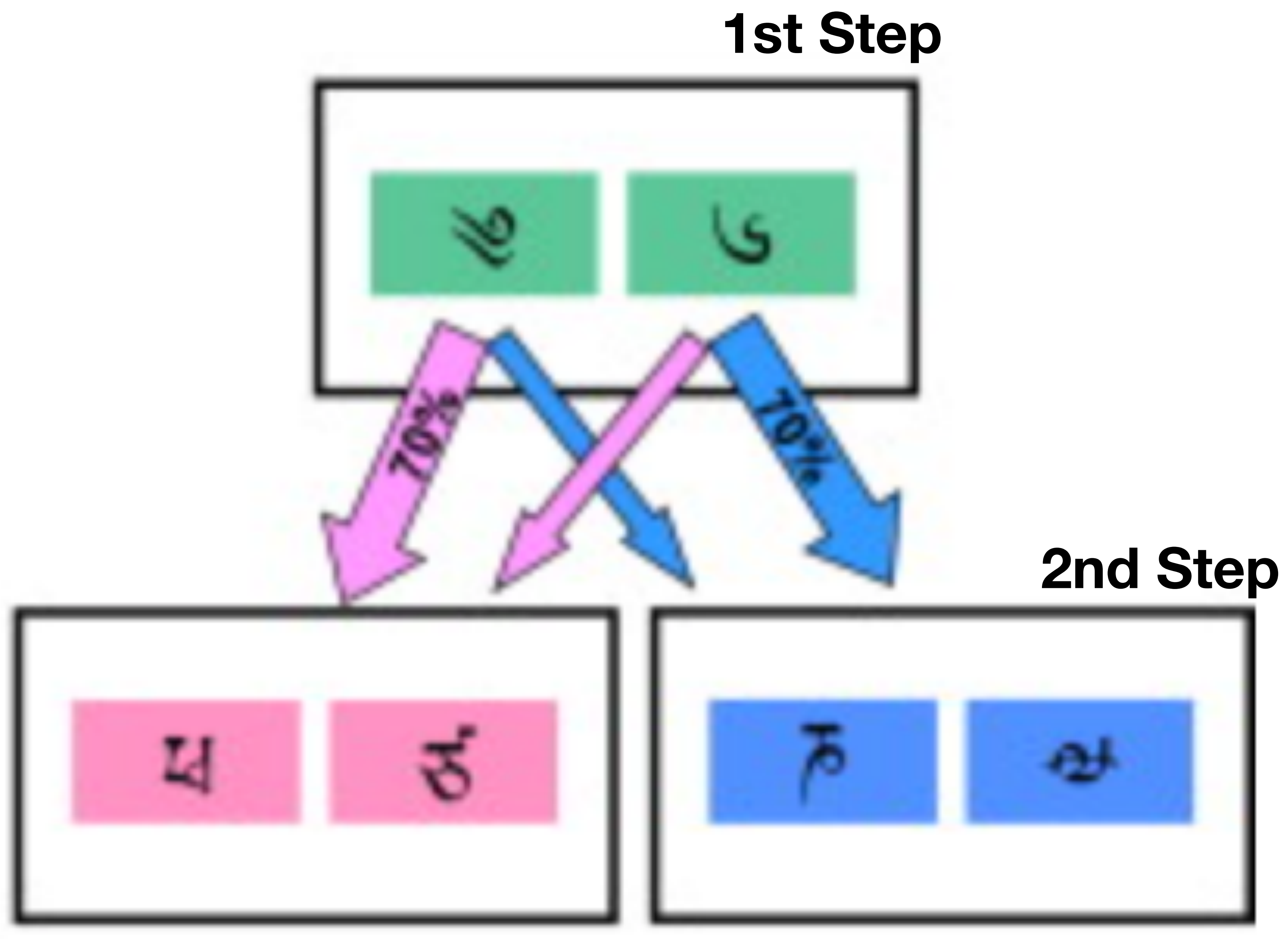

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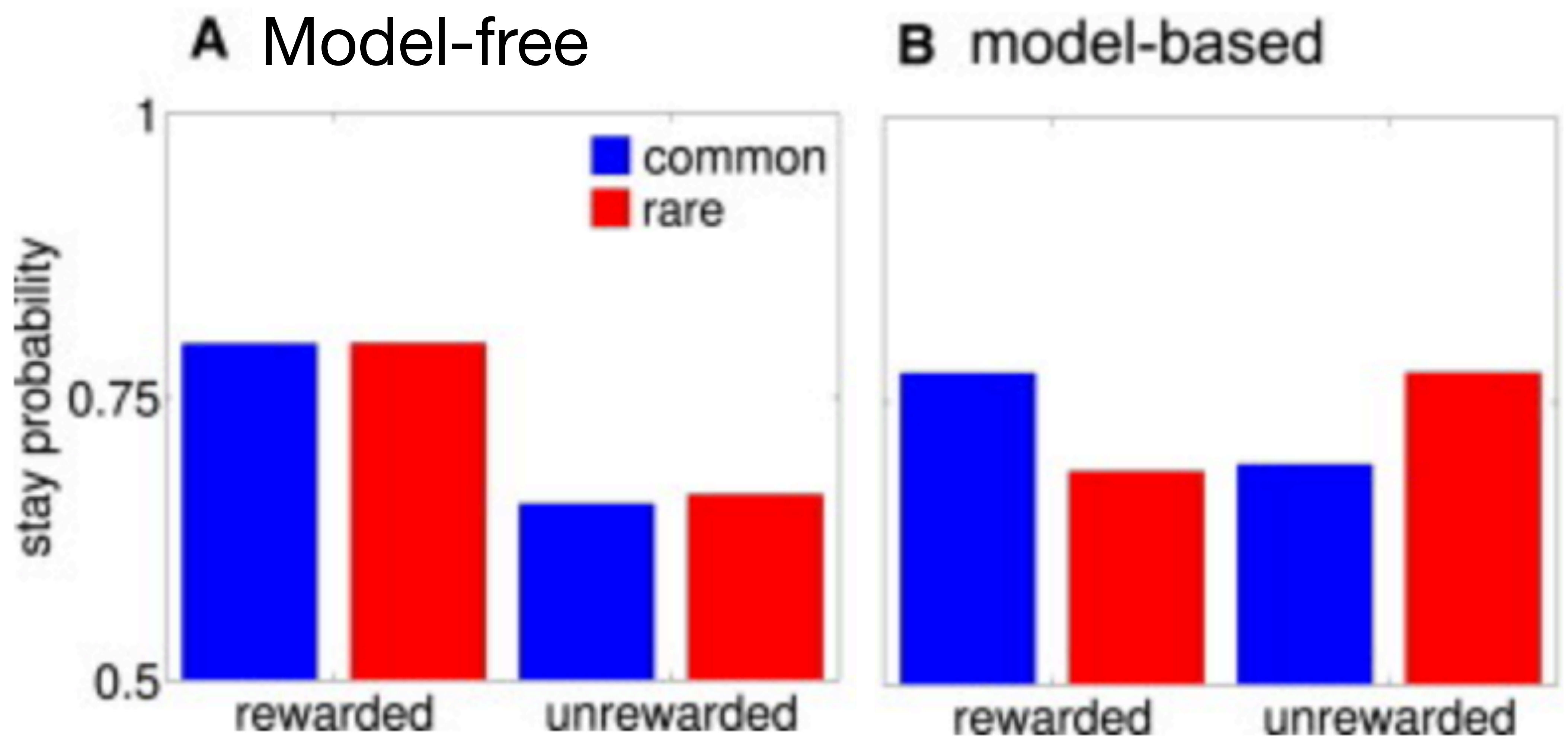

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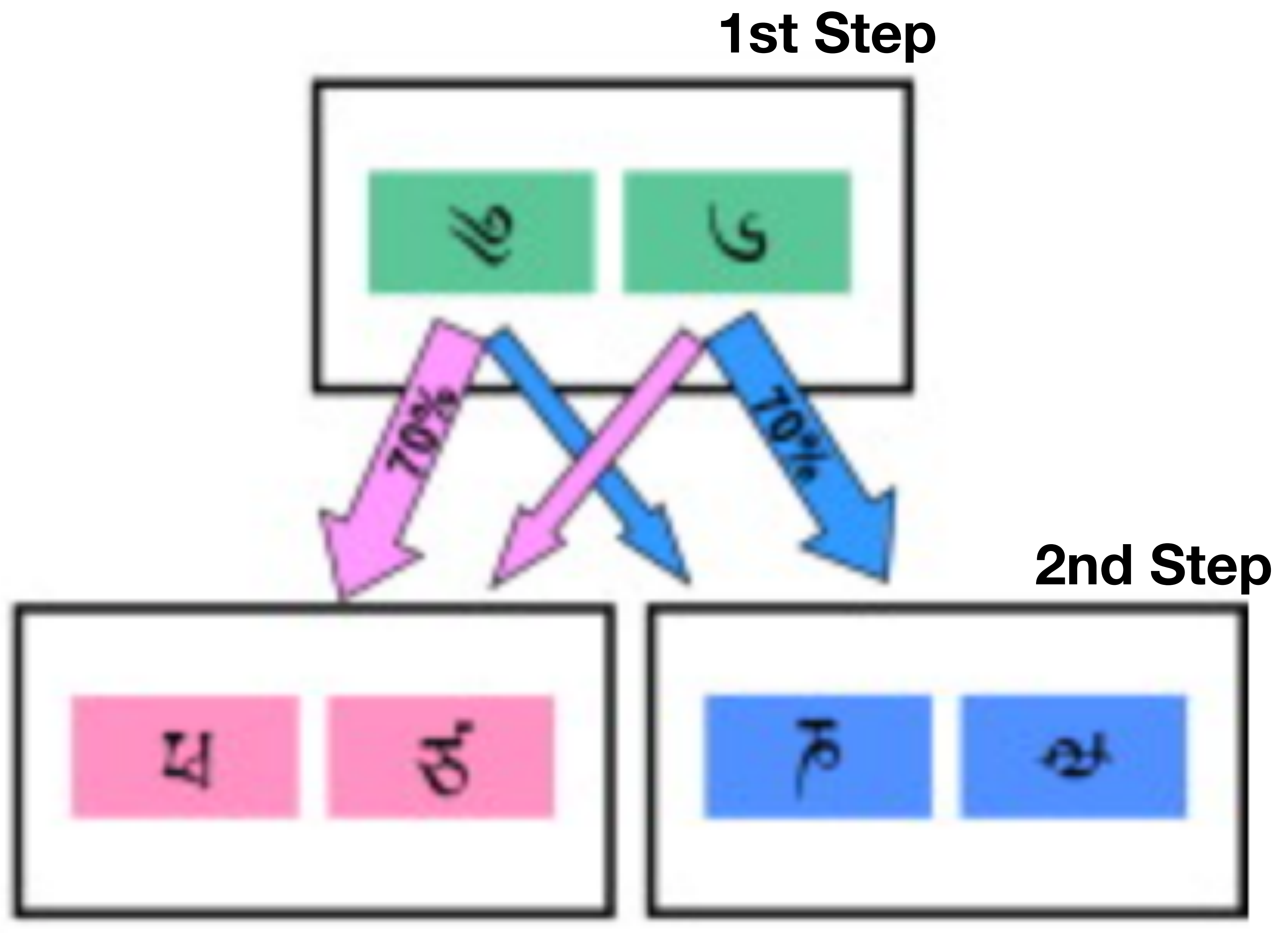
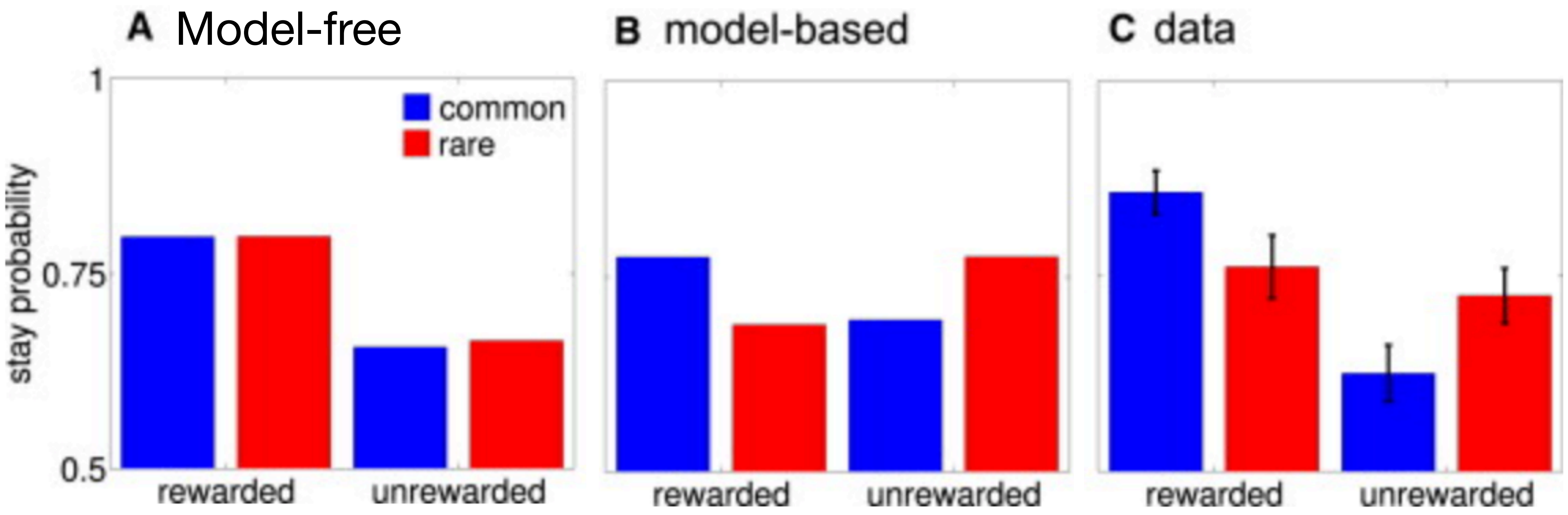


- Model-free predictions depend solely on reward
- Model-based RL predicts different responses depending on common vs. rare
- Data suggests a mixture of both

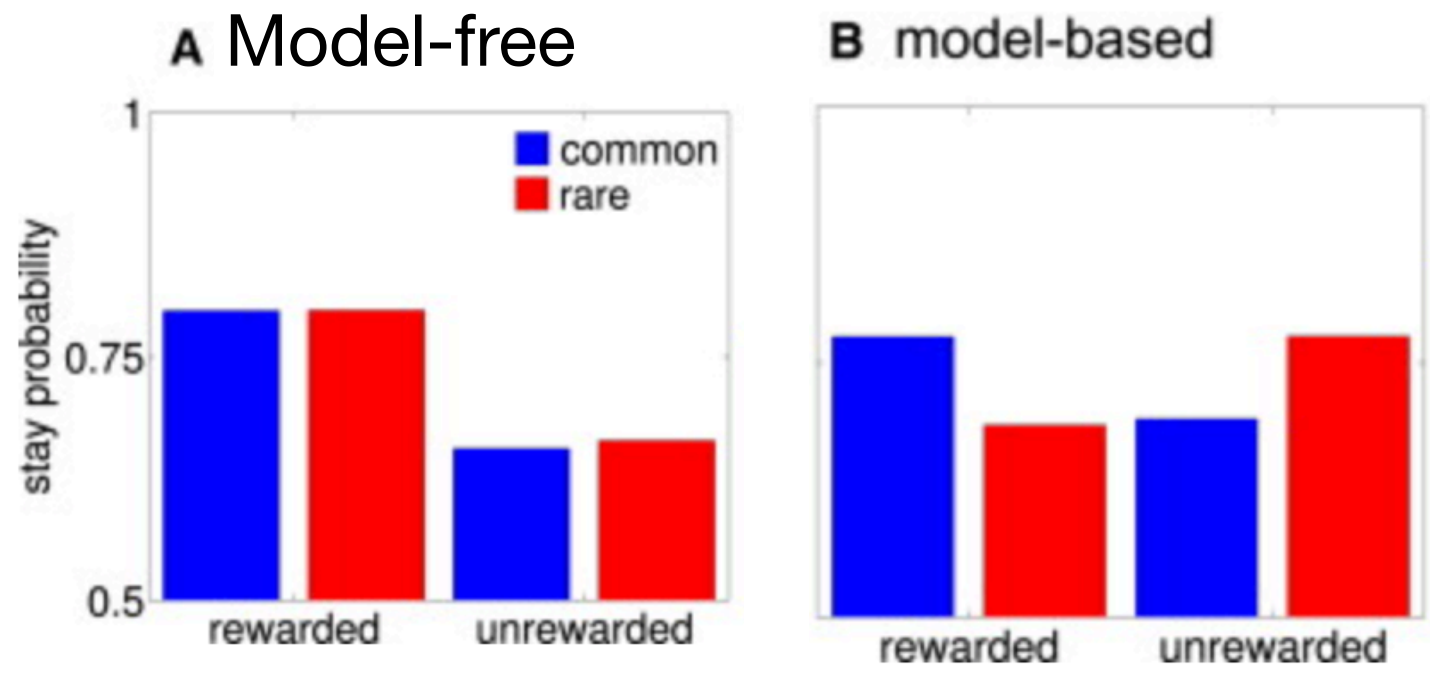


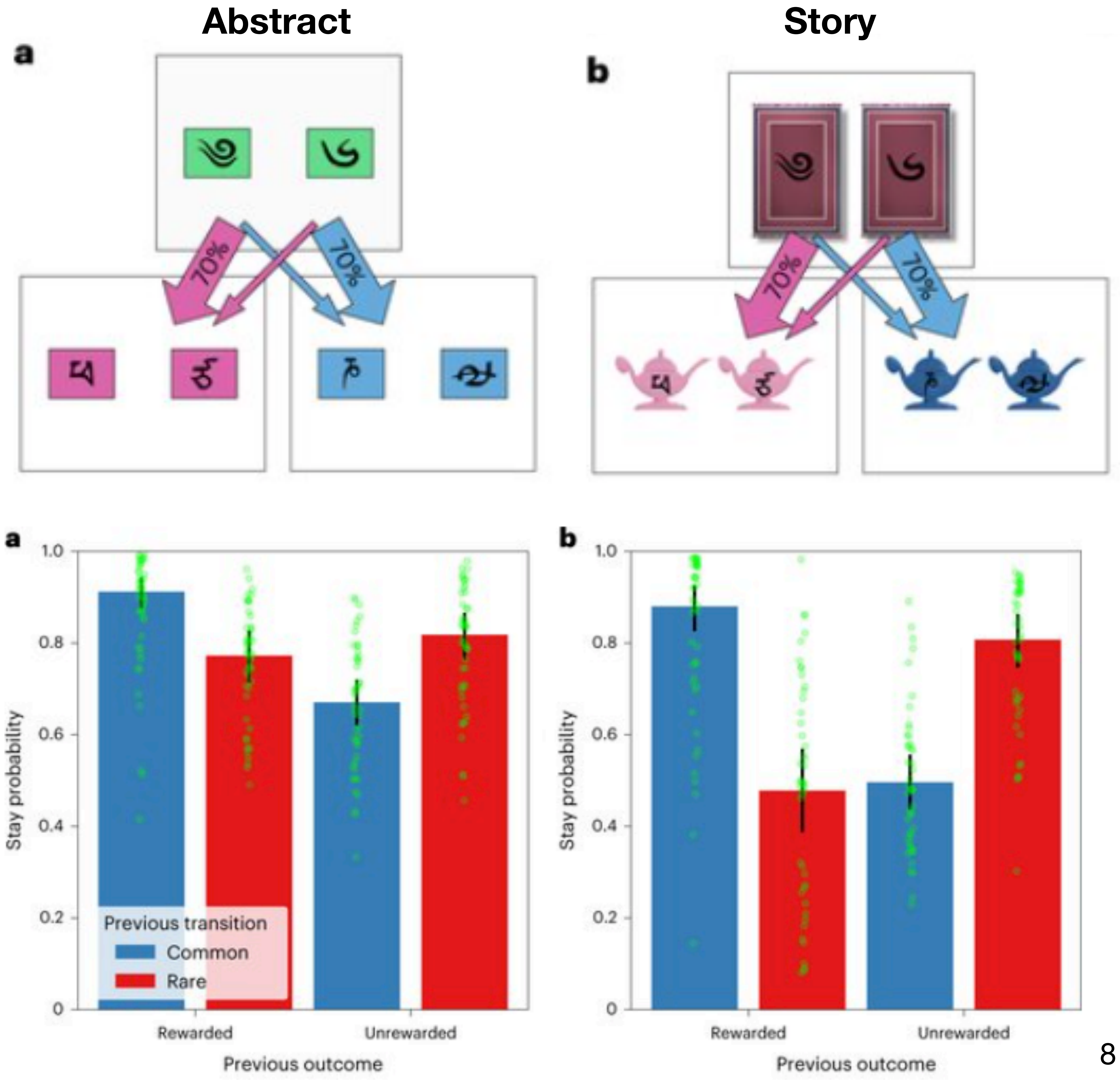
Two-step task (revisited)

- More recent work suggests a different interpretation of that classic result from Daw et al,. (2011)

- Abstract vs. story condition to manipulate how easily it is to understand the nature of transitions

- Story condition was almost perfectly aligned with model-based predictions





Two Pathways for Learning

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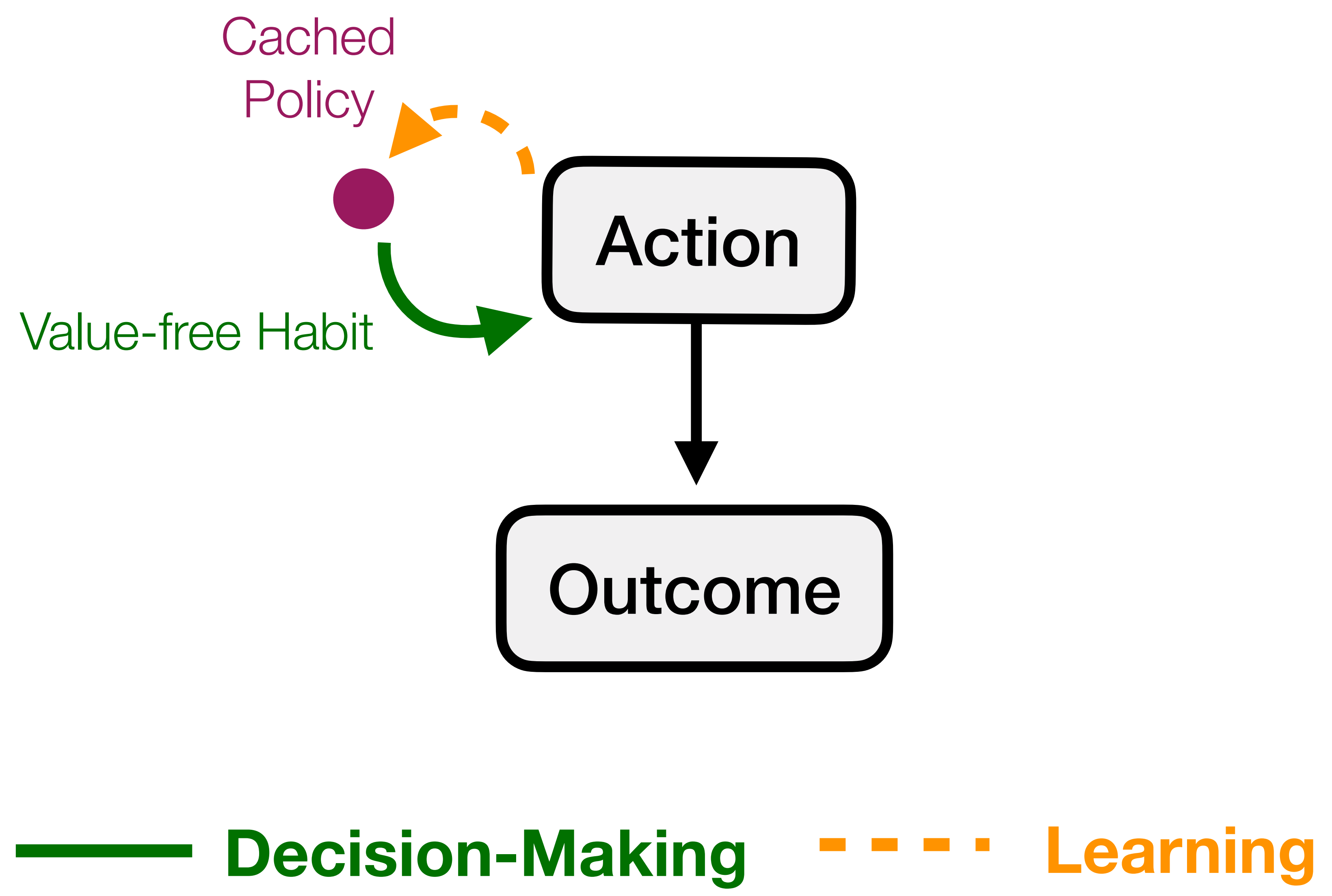
- **Law of Exercise**: Repeat actions performed in the past by learning a cached policy (independent of reward)



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- **Law of Exercise**: Repeat actions performed in the past by learning a cached policy (independent of reward)
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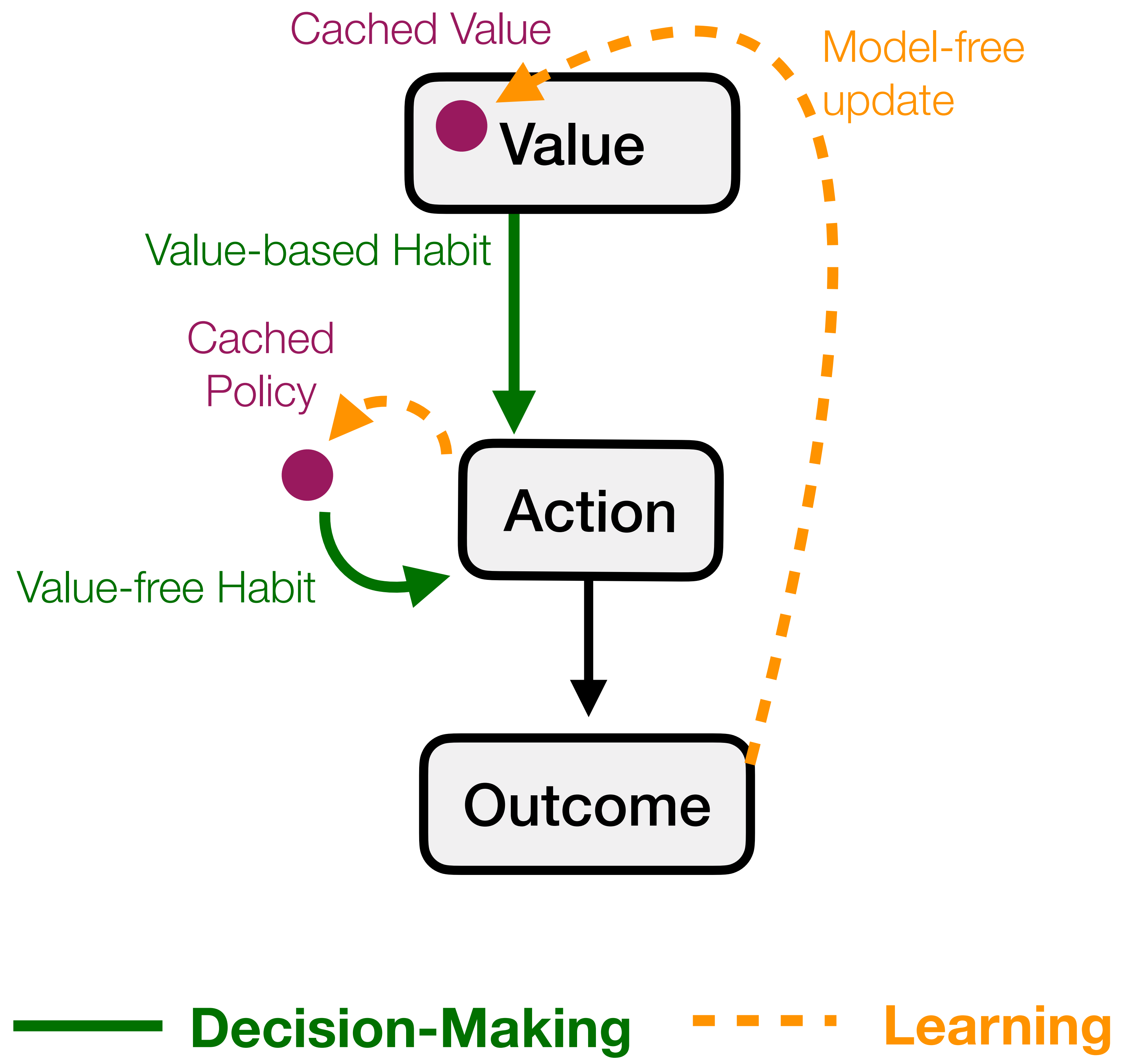

Wu, Veléz, & Cushman (2022)

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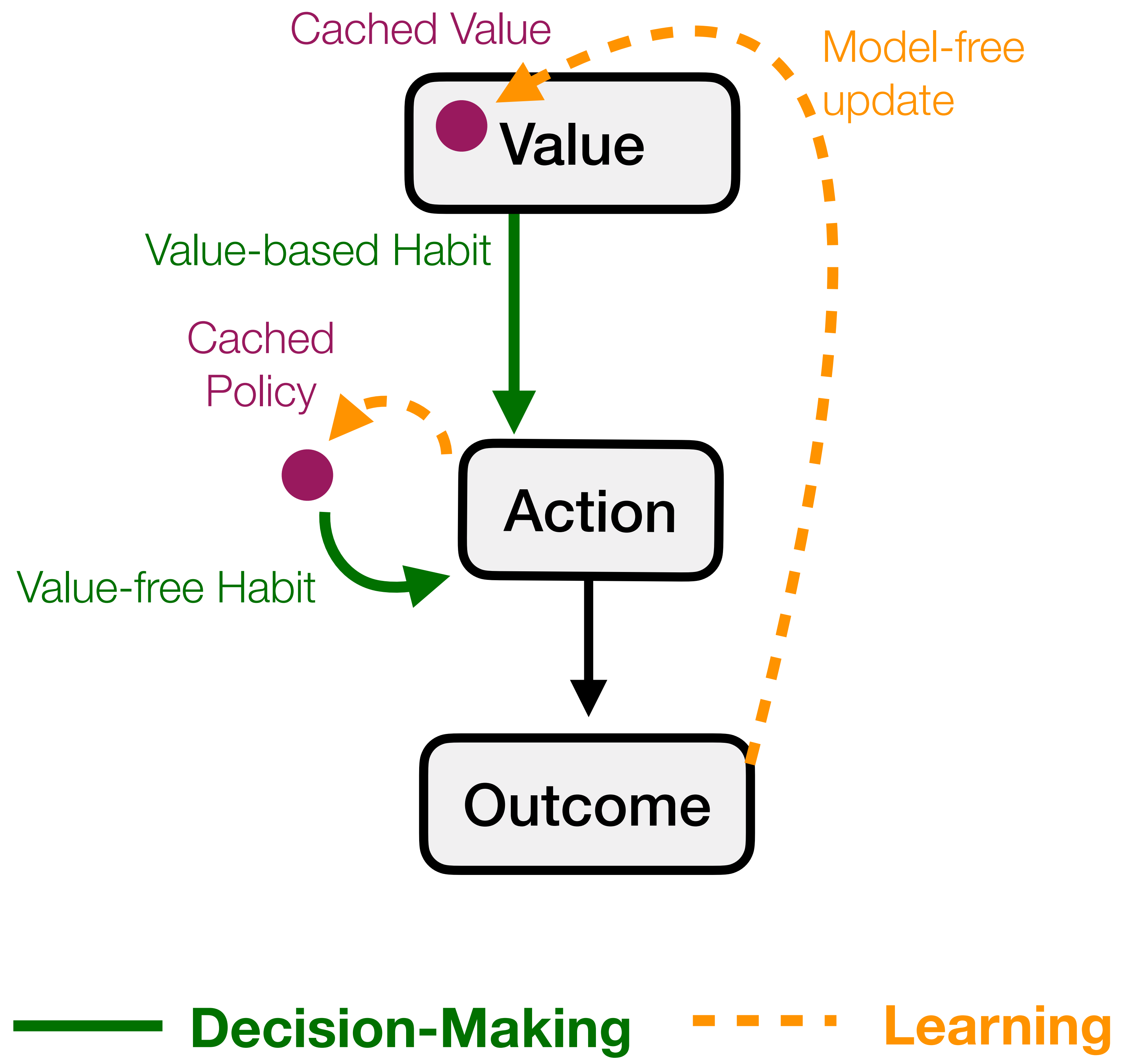

Three
~~Two~~ Pathways for Learning

Wu, Veléz, & Cushman (2022)

- **Law of Exercise**: Repeat actions performed in the past by learning a cached policy (independent of reward)
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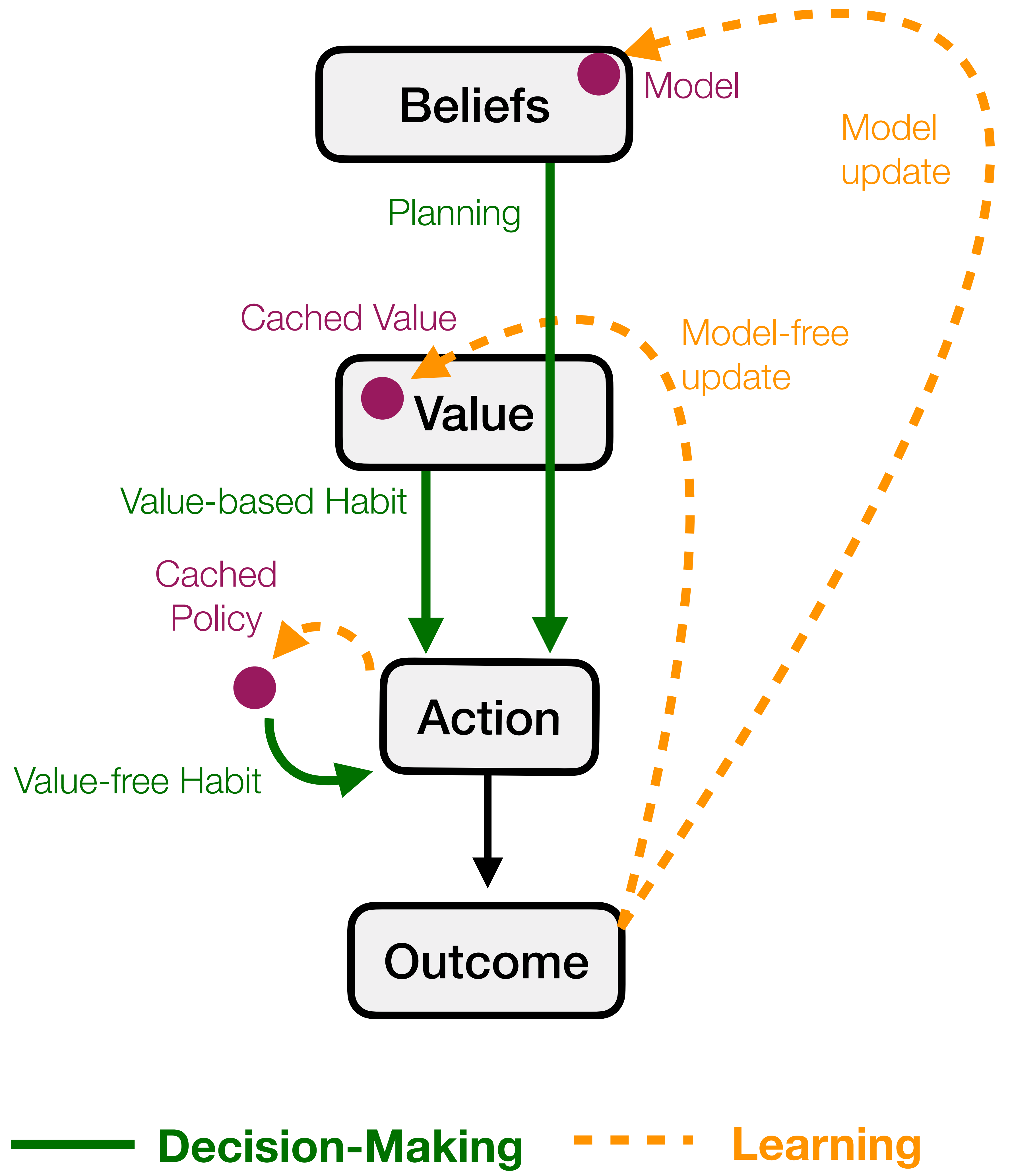

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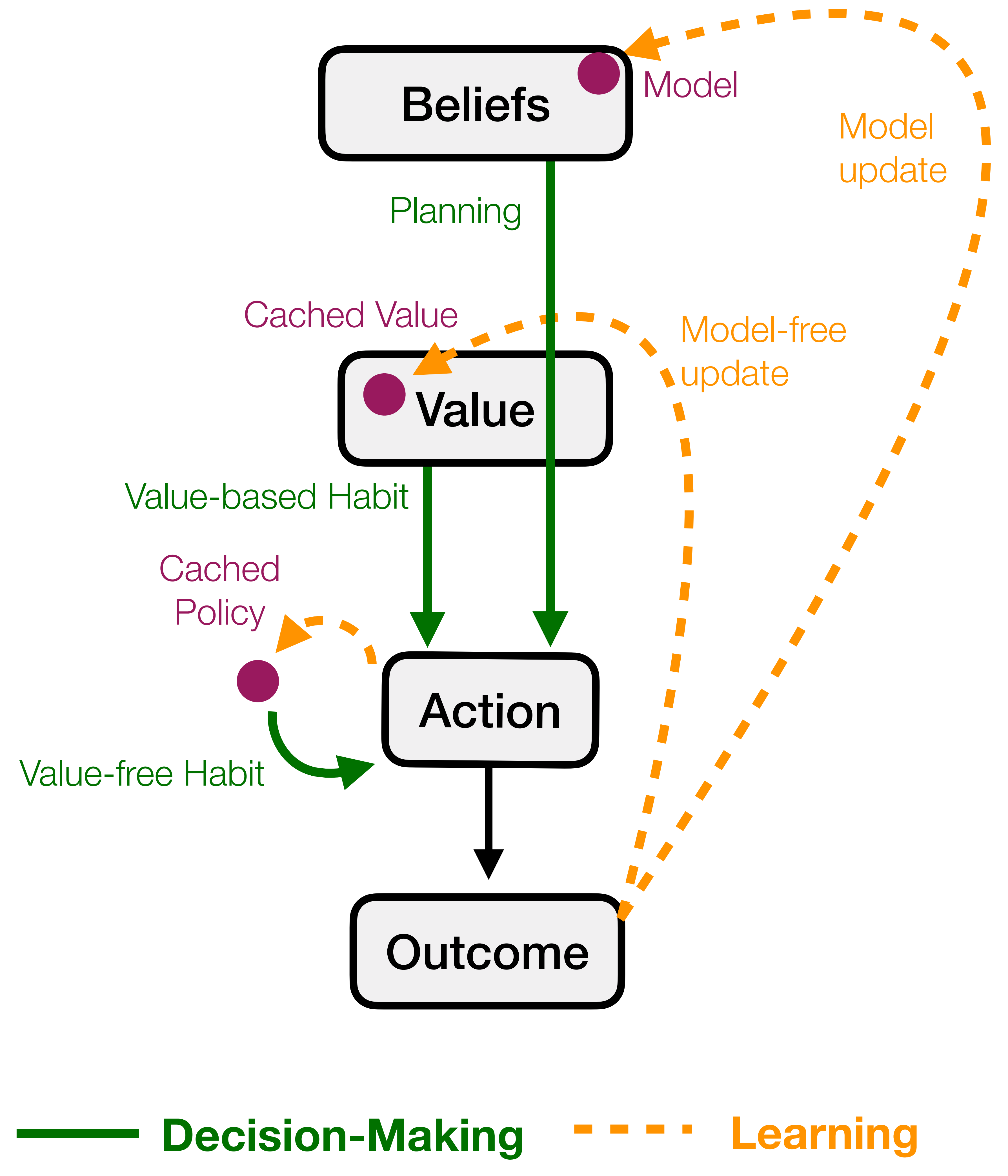

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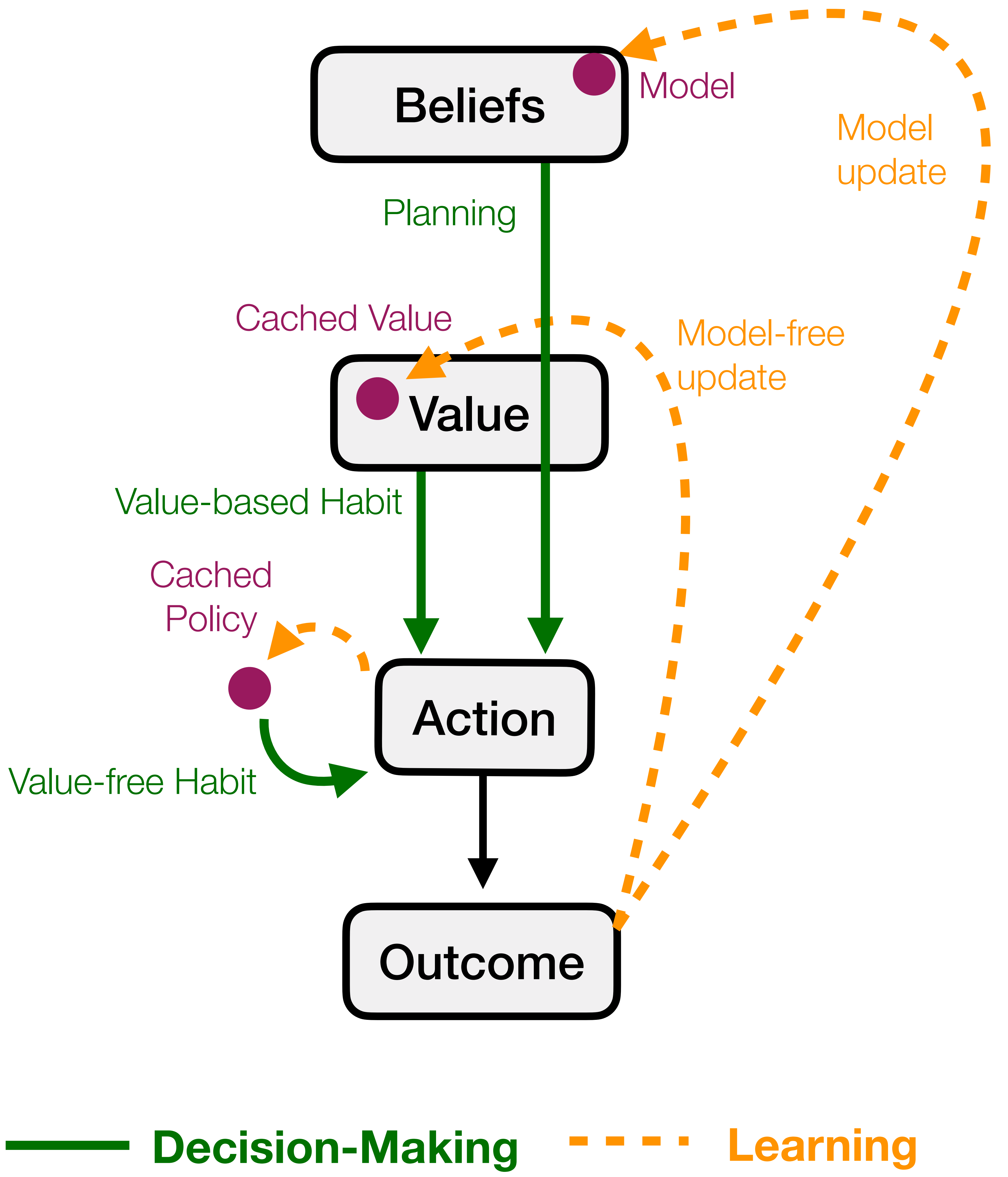
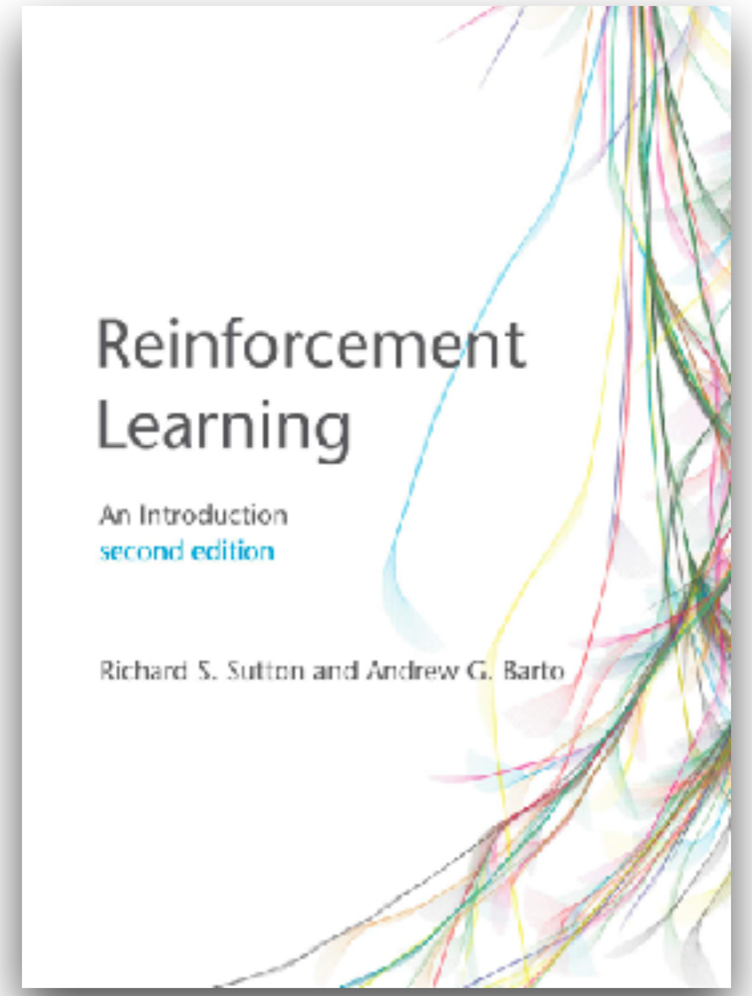
Different pathways not always in competition, but can inform one another! Model-based planning builds better habits!



Model-free vs. Model-based summary

- Computationally cheap to use **model-free learning**
  - Maps onto habits and S-R learning
- Costly but potentially more impactful to use **model-based learning**
  - Maps onto goal-directed and S-S learning
- Model-based learning can help train model-free value functions and policies
- Still open questions about how model-based representations are learned and used in humans (find out next week!)

Further study

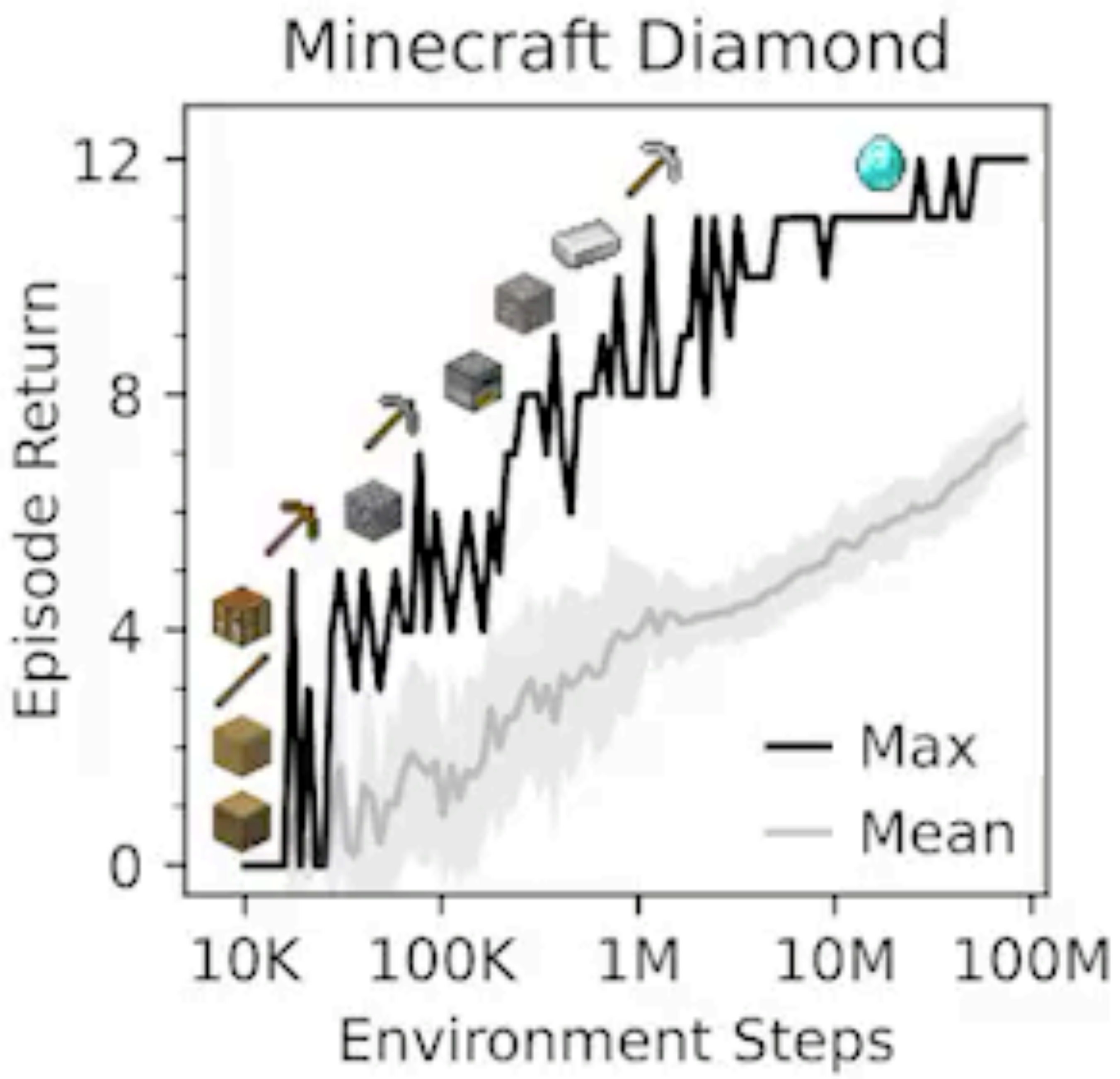


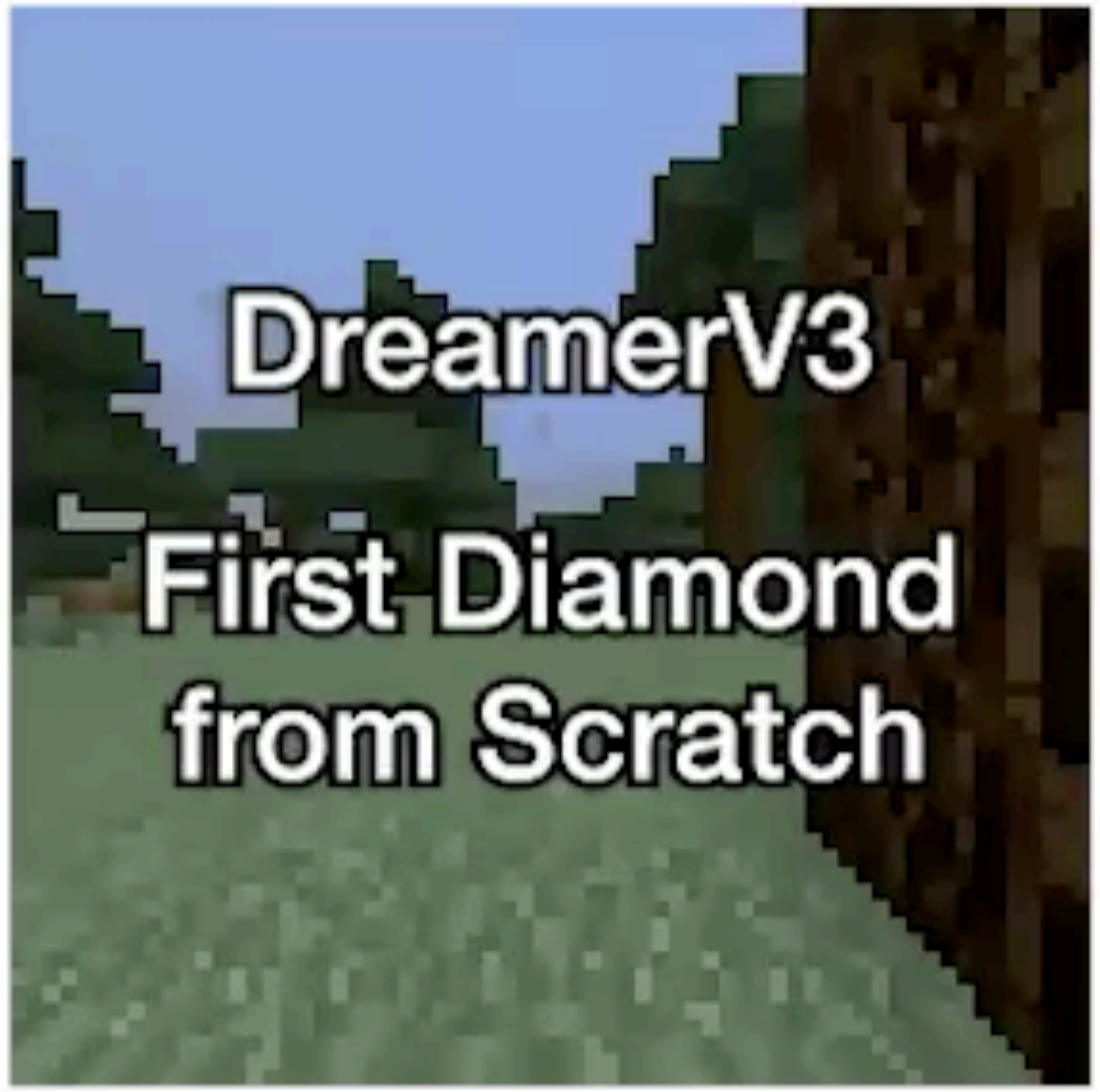
Sutton & Barto book (free PDF link)

R code notebooks for using RL models (with a focus on social learning)
https://cosmos-konstanz.github.io/materials/

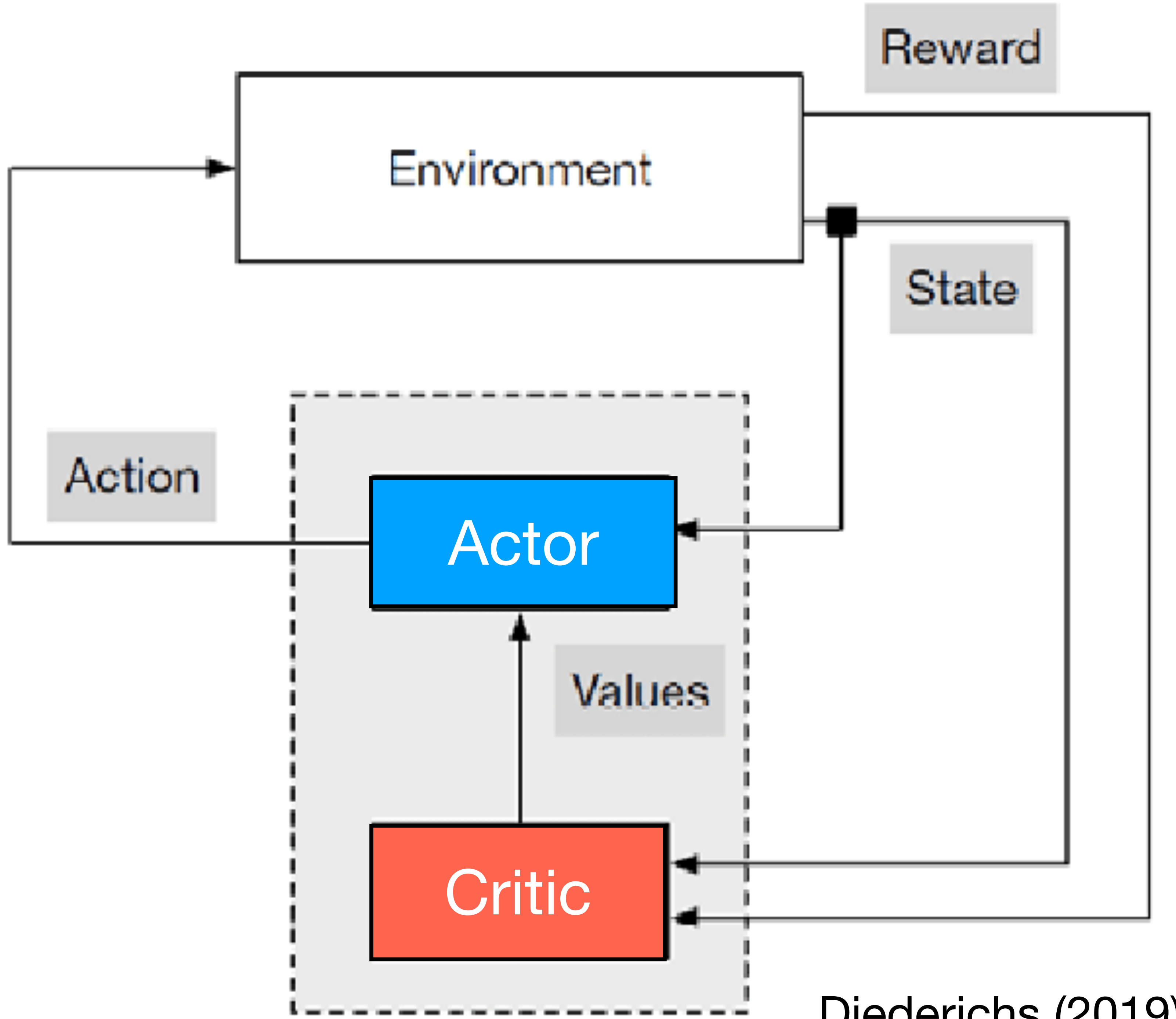
Python tutorial from Neuromatch academy
https://compneuro.neuromatch.io/tutorials/W3D4_ReinforcementLearning/student/W3D4_Intro.html

Next week
Advances in RL





Haffner et al., (2023)



Diederichs (2019)

Stachenfeld (2018)

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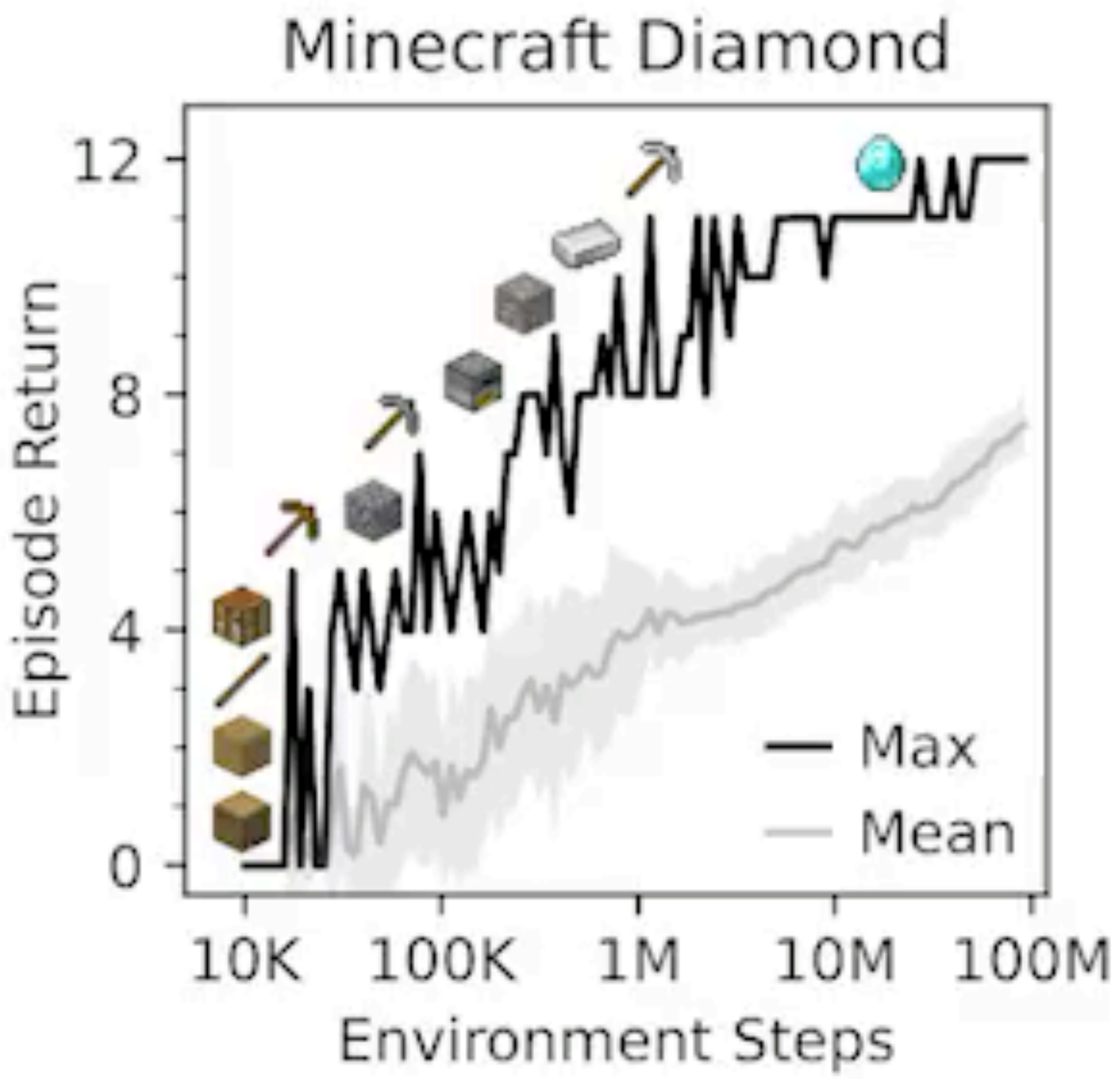

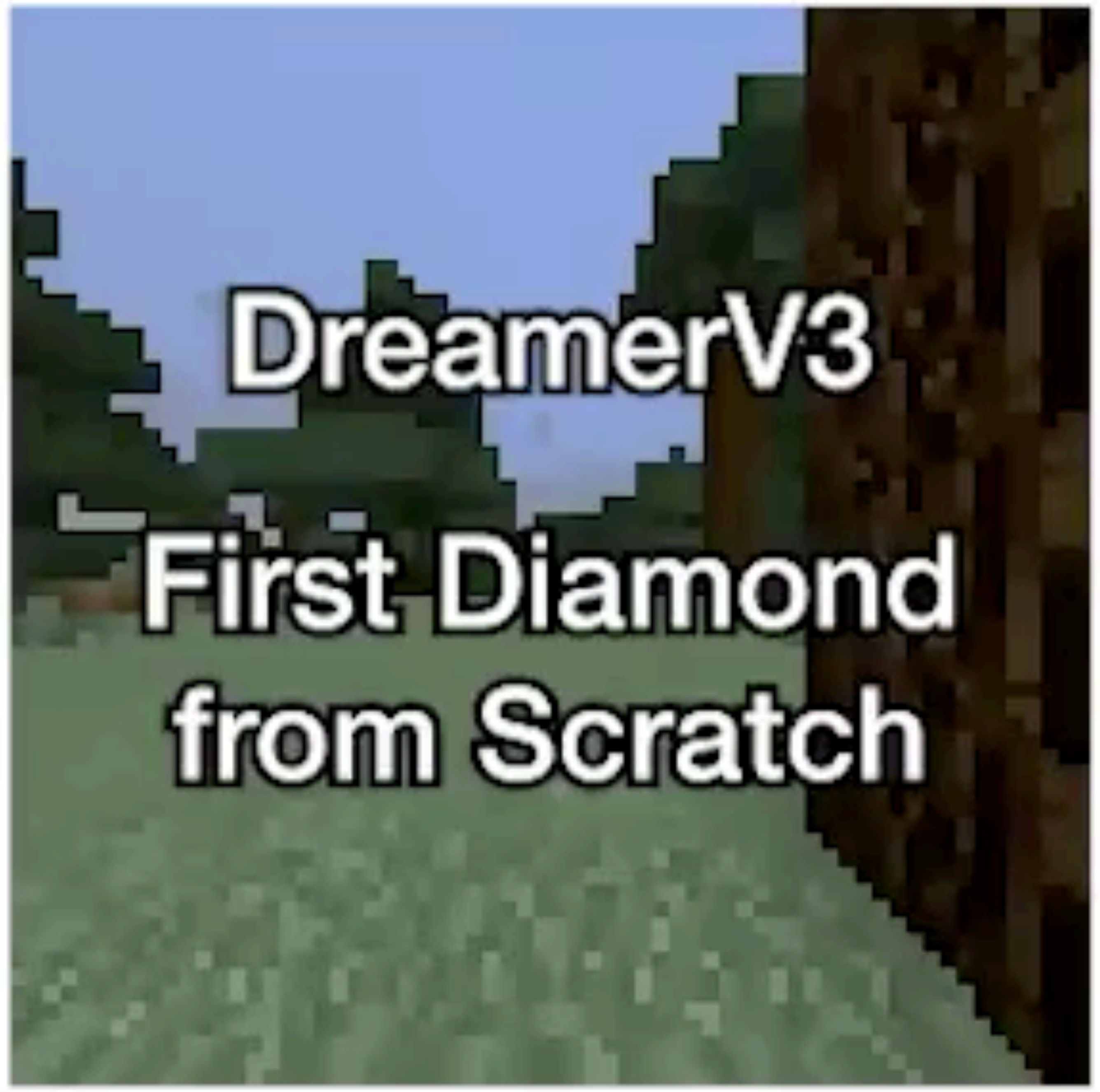

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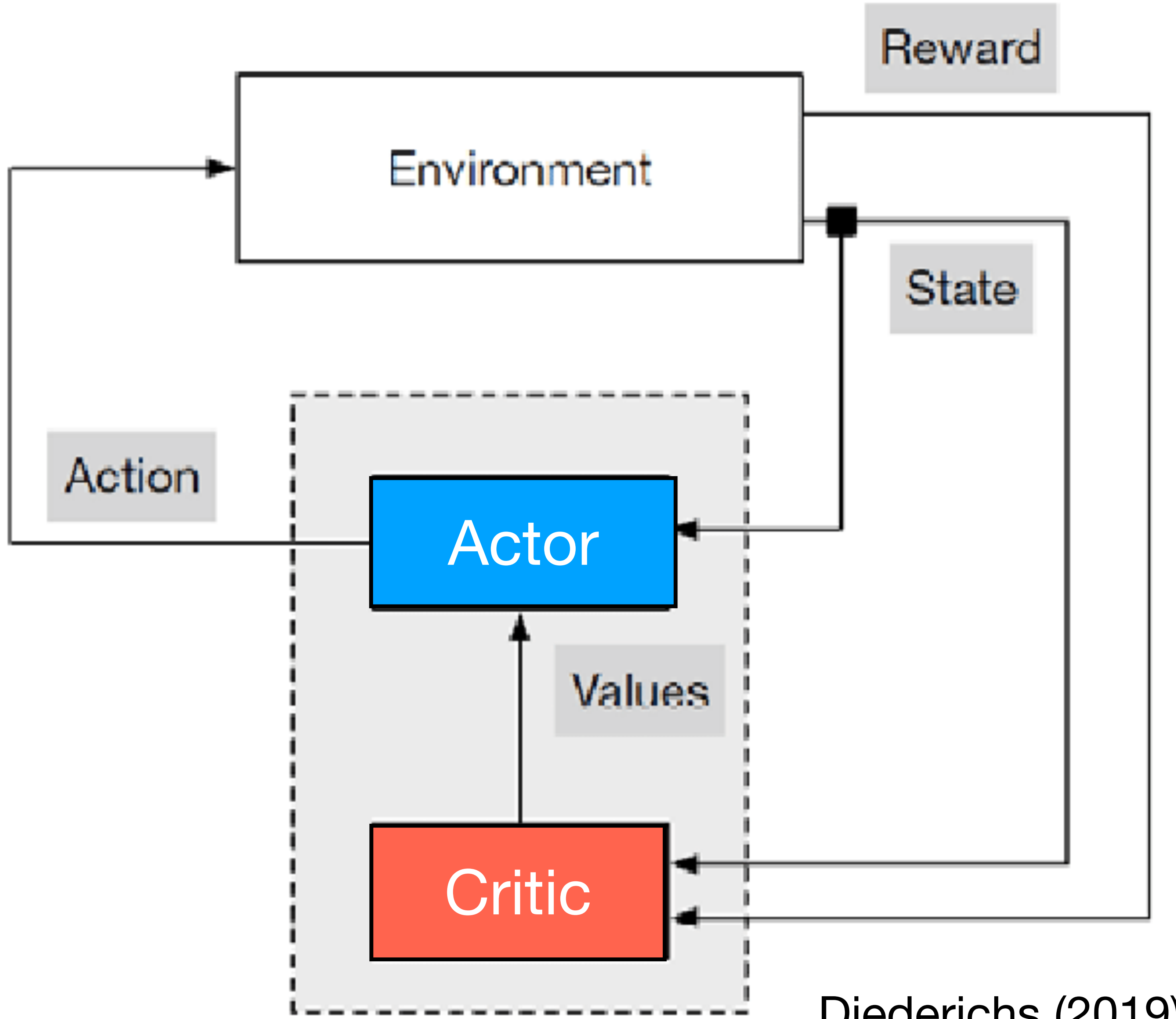

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